REPRESENTATIONS AND PARTICIPATION: LINKING MATHEMATICAL AND SOCIAL RELATIONSHIPS

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This paper explores the idea that intertwining significant mathematical relationships with classroom social structures might support the development of shared understanding, particularly by using the social as both a scaffold for drawing attention to, and a resource for building on, the mathematical. In the case described here, students' affiliations with different representations in the context of their respective contributions to collective classroom discourse helped to resolve a mathematical debate, and to build a shared insight into the relationships among multiple representations of a function.

Students' mathematical activity in classrooms is certainly always tied to social relationships; with luck, a reasonable share of their classroom social interactions will involve the joint exploration of mathematical relationships. This paper seeks to explore intersections between social and mathematical structures in classroom activity, with a particular eye toward the ways those intersections might constitute resources for supporting student learning. Following a brief treatment of relevant theoretical perspectives, I describe a study conducted in a classroom where the social and the mathematical were provocatively and productively intertwined, and present a detailed analysis of one exemplary lesson.

Theoretical Framework

The perspective developed in this paper draws heavily on two other accounts of the relations between social and mathematical practice. Yackel and Cobb's (1996) notion of sociomathematical norms provides a powerful framework for characterizing the points of intersection between students' participation in classroom interactions, and in mathematical activity. On this account, classroom communities develop and maintain collective, local guidelines regarding appropriate ways to contribute to mathematical discussions. Students' understandings of these sociomathematical norms serve as resources through which they come to participate with increasing autonomy in classroom inquiry. In a related way, Stroup, Ares and Hurford's (2005) work on networked devices explores the ways mathematical ideas can structure classroom social activity. They argue that the individual and group-level mathematical objects that students can create in a device network support new forms of classroom participation and collective mathematical activity. The dialectic between these new mathematical and social spaces opened up by generative designs for networked activity provides a resource for re-imagining and transforming mathematics teaching and learning.

Each of these perspectives relies on similar insights into the potentially productive links between the social structures of classroom interaction and the normative and conceptual dimensions of classroom mathematical activity. The premise of this paper is that in certain cases, those links can also serve as resources not only for organizing or reshaping mathematics pedagogy, but also for making important mathematical relationships and structures salient in ways that might support rich opportunities for learning. In particular, I take up the notion of multiple linked function representations (Kaput, 1989) as a set of conceptual objects to be distributed among multiple learners in classroom mathematics activity in order to foster both collaborative mathematical discourse and meta-representational insight. Put more succinctly, I explore the conceptual consequences and the instructional potential of linking different students with different representations of a

collectively engaged mathematical function. Importantly, Yackel and Cobb's account of sociomathematical norms and Kaput's work on multiple representations reflect different theoretical perspectives on learning. While the former draws on a sociocultural framework, in which learning is characterized by students' changing participation in relation to the mathematical practices of a classroom community, the latter emphasizes students' developing understanding of mathematical concepts. Greeno (1997) has proposed an approach to integrating these sociocultural and cognitive accounts of learning; Sfard (1998) likewise argues that these perspectives are more productively viewed as complementary metaphors than incommensurable theories. In a similar spirit, this paper attempts to integrate these perspectives by examining the ways social and mathematical relationships overlapped in one classroom.

Methods

This paper draws on a detailed examination of a single episode in a high school mathematics classroom. Students were 11th and 12th graders participating in the fourth year of a reform mathematics curriculum. As part of a larger study, this classroom was videotaped daily for an entire year of instruction. Observation of classroom sessions, review of the daily videos, and formal and informal interviews with the teacher and students all contributed to the development of an account of learning and teaching practices and classroom norms in this setting. The episode presented in this paper was selected as particularly illustrative of a classroom sociomathematical norm related to the linking of participation and representation during student contributions to discussion, and for a novel moment of collective classroom discovery emerging from that linkage. This episode took the form of a whole-class debate, lasting nearly 30 minutes, about the relationship between the functions $y=x^2$ and $y=x^2+1$. All audible student and teacher contributions to this discussion were transcribed, and all mathematical work either written on the board or projected on an overhead display from a graphing calculator was likewise captured from the video record. Together, these data were analyzed to examine the ways each instance of student participation in this episode involved the use of one or more representations (symbolic, tabular, graphical) of a relevant function in order to contribute to resolving the mathematical problem under discussion. This analysis included coding each such student contribution with regard to the representation on which it relied most heavily, along with more detailed examinations of the mathematical features of those contributions as they built on or refuted those of other students, and of the gradual process through which the class moved from a mathematical dispute to a collective interpretation.

Analysis

A key feature of this teacher's instructional approach involved regularly asking students to demonstrate solutions, pose conjectures, explore ideas or advance arguments through brief and impromptu presentations from the front of the room, usually through the use of either dry-erase markers on a white board or a projected graphing calculator display. One sociomathematical norm governing these student contributions to collective problem-solving discourse involved the use of different representations as resources. Students were encouraged to draw from an array of representations, and some even came to identify themselves in relation to a particular representation—so that a student might even explain for their choice of representational mode by saying things like: "I'm a graphing person."

Linking individual participation with representation in this way appeared to serve as a classroom resource both for resolving mathematical debate, and for developing collective understanding of the links among multiple representations. In the episode examined in detail for this paper, 11 students made a total of 13 successive presentations from the front of the

room regarding the problem under discussion. Of these contributions, summarized in Table 1, 7 were coded as relying primarily on graphical representations, 3 on tables, and 3 on symbolic expressions. The prominence of graphical representations reflected the nature of the debate; some students thought that the graph of $y=x^2+1$ had a "different shape" from $y=x^2$, while others asserted that the two functions were identical but for a vertical shift. The use of a public graphing calculator display in some students' presentations only strengthened the support for the incorrect interpretation, and resolving the debate involved a careful weaving among the symbolic, graphical and tabular perspectives by several students in order to reconcile the correct interpretation of the vertical translation with a confusing graphical display.

Student	Contribution	Representation
1	Added one to values in Table, "makes it steeper"	Table
2	$y=x^2+1$ can't touch x-axis because $0^2+1=1$	Symbolic
3	f(0)=0+1 so graph through origin incorrect	Symbolic
4	Erases graph that looks least like x^2—too skinny	Graph
5	"2^2+1…doesn't change the shape, it just changes the height."	Symbolic
6	Graphs both functions on calculator, argues they're different	Graph
7	Graphs x^2+10	Graph
1	"it's almost the same shape, but it's…compressed some"	Graph
8	"it's the same exact graph" "all the y-coordinates are up one"	Table
9	"it only looks squinched on the calculator"	Graph
10	Compares differences in table	Table
11	"If you just draw it by hand… they stay just one apart."	Graph
7	Traces projected curve, moves projector	Graph

Table 1. Student Presentations and Representations

In the analysis that follows, excerpts from four of these student presentations will be examined in greater detail—one each from the symbolic, graphical and tabular modes, followed by an argument that attempts to make connections between multiple representations.

Symbolic

Students used symbolic representations to make two key points. The first was that unlike $y=x^2$, the graph of $y=x^2+1$ could not touch the x-axis. The second point was that because the two functions differed only by a constant, their respective graphs had identical shapes. Student 5 elaborates the latter argument in this excerpt:

Student 5: The thing that makes it this shape, is, um, would be the x squared [indicates expression]. It's the x-squared part. Because when you do, like, if this was two [touches x-axis at x=2], it's four here [draws the point (2,4)], and negative two is negative four here [draws the point (-2,4)]. So it's the same shape, and then if you have a negative, er, two squared plus one [indicates expression again], it doesn't change the shape. It just changes the [moves hand up and down to show vertical translation]...the height.

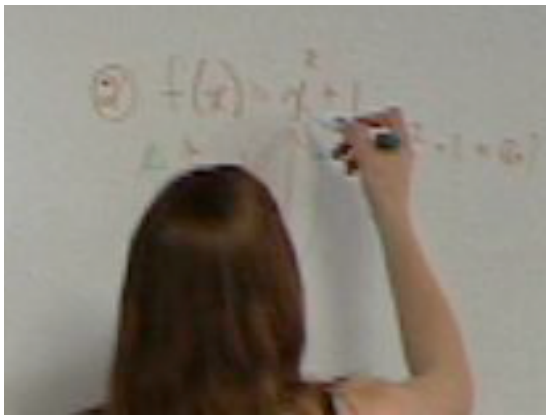

Figure 1. Symbolic representation

Importantly, this student illustrates her argument through repeated references to a graph, including drawing two points on the board. But I have coded that argument as symbolic because she relies on references to the algebraic expression as the evidence for her assertions about that graph. In particular, she stresses that "it's the x squared part" of that expression that "makes it this shape," so that adding one "doesn't change this shape."

Graphical

The next student to come up disagreed with that conclusion, and illustrated his counterargument by plotting several points associated with the two functions, sketching the graphs, and then asserting that this work "shows that it's different". The sketches failed to provide clear support for or against his claim, so the teacher suggested that he graph both functions on a calculator projected onto the board. This prompted an extended class discussion about the appropriate window dimensions in which to view and compare the graphs. After adding the function $y=x^2+10$ in order to provide a clearer contrast, they eventually settled on the display shown in Figure XX. The following excerpt illustrates the way several students interpreted this graphical display:

Student 1: I think it's like, it's almost the same shape, but it's kind of like, compressed some, so like, even if it's way, like if you have another one way up here, like x squared plus like 20, it's gonna go out to the same thing.

Student 7: Right, but the…

Student 1: Like the same lines. Like, all three will go up eventually, looking like really close.

Teacher: So you're saying, now, I'm going to rephrase what he's saying, and you tell me whether this is correct. You said that eventually it's going to look like that, but at the bottom, it's scrunched differently.

Student 7: Yeah.

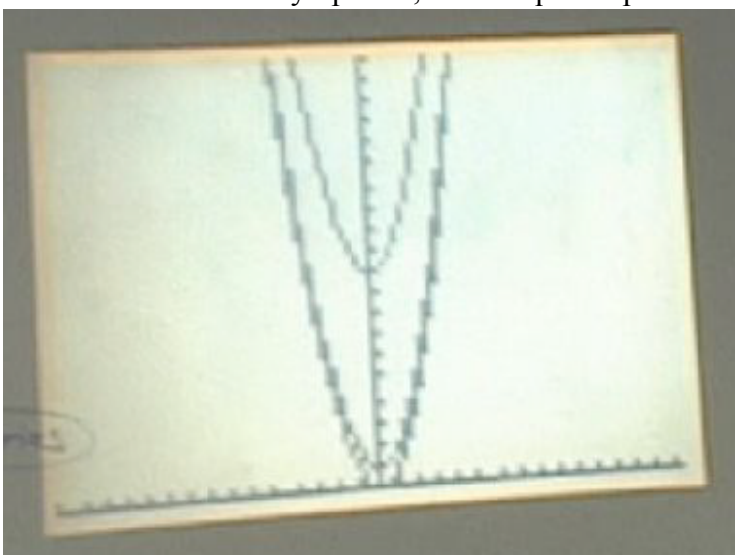

Figure 2. Graphical representation

These students agreed that while these parabolas have "almost the same shape," and share similar end behavior—they "go out to the same thing," they were "compressed" or "scrunched" differently at the bottom. This misunderstanding of the relationship between the functions appears to be a consequence of a sort of optical illusion produced by comparing the vertical spacing between the curves near the respective vertices with the horizontal spacing at other points.

Tabular

The next student to come up used tabular representations to challenge this interpretation of the two graphs as having different shapes. This presentation opened with an exchange between the teacher and student about the latter's choices of representational medium and mode:

Teacher: Ok. So [Student 8] wants to go up.

Student 8: Oh, not on the calculator, but on the…

Teacher: Ok, you want to talk. Go ahead.

Student 8: Yeah, just on the t-table.

Teacher: Fine.

Student 8: Ok, so for this one, this is…[draws a table of values for $y=x^2$]. And for this one, it's just…[draws a table of values for $y=x^2+1$].

Seated Student: I am a graphing person.

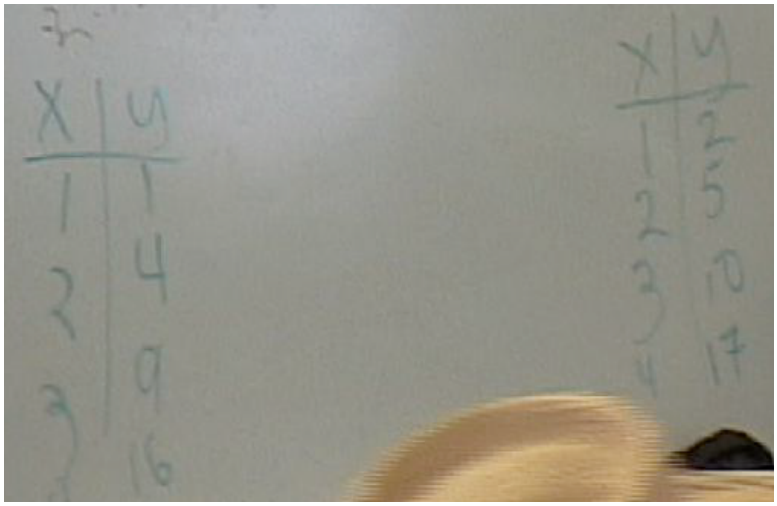

Figure 3. Tabular representation

Student 8: And, it's like, it's the same exact graph, it's just that all the y-coordinates are up one…this two is just up one from one, and this four is just up to five, it's just the whole thing is just up one.

This student expresses her preference for the table rather than the graphing calculator as a resource for making her argument. In apparent response, a student seated near the camera asserted aloud that he was "a graphing person"—a moment illustrative of the classroom sociomathematical norms regarding multiple representations. Meanwhile, Student 8 used the tables she had drawn to demonstrate that the second function was indeed just a vertical translation of the first—"the whole thing is just up one."

Linking Representations

Presented with these symbolic and tabular cases for the two functions having the same shape, and the apparently clear graphical demonstration of their difference, the class remained divided about the correct interpretation. The next excerpt proved pivotal in resolving that division, and I will argue that it achieved that resolution precisely because it involved drawing connections among multiple representations:

Student 11: If you just draw it by hand, you see that it's the same, like, from the little table, because, like, if you just do it yourself, you just go, like, one, two, three, four [counts off as he scales and labels x-axis]…And then up one, two, three, four, five, six, seven, whatever [scales y-axis]. And then you just, like, here it's, so you know it starts here, and then like, on the first one you're here [makes a mark on the graph at (1,1)], and then you're at four [marks the point (2,4)]…up here. And on the second one you'd, so you move up here [marks (1,2)], and then you're at five [marks 2,5]. And you just keep going and they're, they stay just one apart [holds his index finger and forefinger a unit apart].

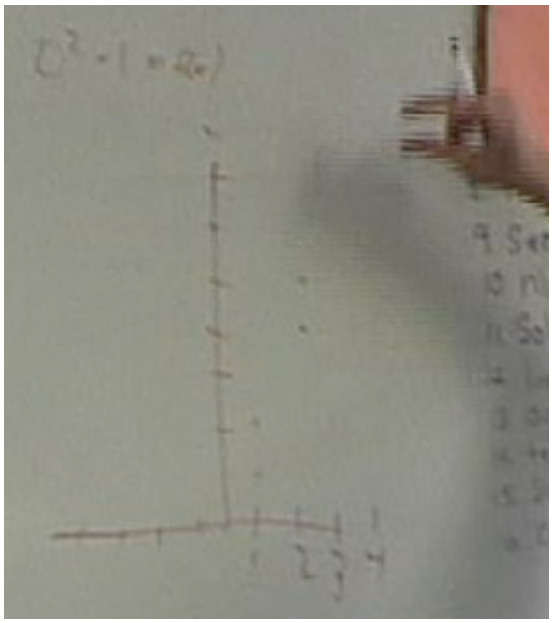

Figure 4. Vertical shift

This student began by emphasizing the connections between the graphical—"if you just draw it by hand"—and the tabular—"it's the same [as] the little table." To illustrate these connections, he plotted the same points Student 8 had identified as ordered pairs in her t-tables for the two functions. Just as Student 8 had stressed that "two is just up one from one, and this four is just up to five," and that "just the whole thing is just up one," Student 11 emphasized that the points from the respective functions "stay just one apart." He then carefully sketched the two curves and again emphasized that there was "just one [vertical] difference all the way up." In doing so, he effectively linked the one-unit difference that his classmates had identified in the symbolic expressions and tables of the two functions to their respective graphs. As student 11 concluded his explanation, the remaining students who had thought the shapes of the two graphs were different acknowledged that they had now been convinced otherwise.

Discussion

This debate was both fueled and eventually resolved by the distinct affordances and limitations of these different representations. Students who drew on the symbolic and tabular modes were able to clearly explain and illustrate the one-unit vertical translation from $y=x^2$ to $y=x^2+1$, but that vertical shift was more difficult to observe in the graphical displays provided by hand-drawn sketches and the graphing calculator. Student 7, in fact, went to the front of the room to add the graph of $y=x^2+10$ to the calculator display because she thought it would help student 6 to see that the curves had the same shape, but concluded after seeing the graph by agreeing with students 1 and 6 that the parabolas were "scrunched differently" at the bottom.

These apparently contradictory interpretations of these different representations of the same functions posed two conceptual problems for the class. First, the class could not, prior to Student 11's presentation, come to agreement about which interpretation of the relationship between the shapes was correct. And second, they could not resolve the apparent discrepancy between what they expected to be equivalent representations. Student 11's linking of arguments from the table and graph provided a way to solve both these problems simultaneously. He gave a convincing answer to the question of whether the two functions had the same shape, and he demonstrated that that answer was indeed supported by both representational modes.

My point here is not to overemphasize the significance of Student 11's presentation relative to those made by other students. In fact, the presentation's effectiveness appeared to be supported less by the clarity of the speaker or the depth of the insight than by the progression of other student contributions on which it built. In other words, the link between the graphical and tabular modes was less a consequence of Student 11's unique perspective than it was an emergent feature of several successive presentations from distinct representational perspectives. While Student 11 made reference to the tables drawn by Student 8, his presentation was clearly focused on the graphical mode—as in every other student presentation, a single representational mode predominated. But by highlighting a connection between his graph and Student 8's table, and by finding a way to interpret the graph that aligned with other students' interpretations of the tables and symbolic expressions—as demonstrating the identical shapes of the two curves—he helped to complete the class's gradual weaving together of these multiple representational threads.

Ultimately, the episode supported a view of the links among multiple representations precisely because each student—in keeping with a classroom sociomathematical norm—stayed in a single representational mode, as agreement among representations emerged as a necessary corollary to agreement among students who had been on opposite sides of a debate. In terms of the dialectic proposed by Stroup et al., the social space defined by this succession

of student presentations with the support of representational tools helped to delineate a corresponding mathematical space. As students brought their respective views of the two functions into alignment, they drew attention to the ways the different representations underlying those views were likewise aligned: the links among multiple function representations were made visible through transactions among multiple social actors. At the same time, the mathematical links among the different representational artifacts invoked by students provided resources for negotiating those transactions.

Conclusion

I conclude by suggesting that this classroom episode may point toward some valuable implications for the design of mathematical learning environments. Work such as Kaput's attests to the promise of multiple linked representations in computer-based learning environments for supporting student understanding of complex mathematical concepts like function. But that promise may be much more likely to be fulfilled if the links among those representations are socially as well as technologically mediated. To that end, my own design-based research efforts are currently devoted to investigating the potential of linking mathematical and social structures as a principle for designing collaborative problem-solving activities that utilize classroom computer networks (White, 2006; 2007). These designs connect the relationships among a set of mathematical objects, including but not limited to multiple representations, with the pedagogical organization of relationships among multiple students. I hope that these designs will not only make rich classroom conversations like the one reported here more common, but also contribute to broader theorizing about the relationship between the social and mathematical dimensions of classroom activity.

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