The Effects of a Proof Mapping Instructional Technique on High School Geometry Students and Their Ability to Write Geometric Proofs

By

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THESIS

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Abstract

Title: The Effects of Teaching Proof Mapping to High School Geometry Students on Their Ability to Write Geometric Proofs

Research Questions:
In what ways does teaching the "proof maps" affect high school students' performance on writing accurate and logical geometric proofs?
Sub-questions:
o How does the student's ability to memorize theorems influence student ability to use the proof map method?
o To what degree will students attempt a geometric proof if they learn the proof map method compared to leaving that test question blank?
o In what ways does using the proof map method affect students' confidence in their own ability to write proofs?
o Are students more inclined to offer help to other students after they learn the proof map method?

Research Activities:
This research investigated the utility of teaching high school geometry students a "proof mapping" technique for writing geometric proofs. "Proof-mapping" provided students with a means to diagram their thinking process while writing a proof. Proof statements were enclosed in circles, and arrows to circled statements were labeled with the appropriate justifying theorem or definition, allowing students to connect related concepts together and to see the proof develop. This study took place in a comprehensive high school in a suburban northern California city. The focus group for this study was a high school geometry class (n = 32) with a diverse population including multiple ethnicities and students of various socio-economic statuses. This intervention lasted 6 weeks. Students were taught the proof mapping method and utilized it in group work, pair work, homework, and daily warm-ups as well as on tests and quizzes. These activities took place 2 to 3 times a week. Students were also given a pretest, post-test, pre-survey, and post-survey to track changes in ability and attitude toward proofs. Teacher observations tracked student involvement helping their classmates. During proof writing,

this method provided opportunities for discussion between the students and teacher. Students' thinking processes and common errors were revealed. Students seemed to enjoy the process more than the direct two-column proof and were more successful at writing proofs by the end of the intervention. The method provided a means of communication about proofs through a visual diagramming process, encouraging an open-ended discussion about how to think strategically about writing a proof. Although the students did improve at generating proofs, additional benefits included increased discussion about proof writing strategies and students' confidence in their ability to write proofs.

Grade: Secondary

Research Methods:
Observation-Student engagement/behavior tallies; Questionnaire; Student work; Survey-Attitude; Survey-Self-assessment; Teacher-made assessment

Curriculum Areas:
Math-Geometry

Instructional Approaches:
Assessment/Evaluation; Class discussion; Collaboration/Teaming; Conceptual understanding; Graphic organizers/concept maps; Homework; Math-Teacher feedback; Vocabulary development; Test revision

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The Effects of a Proof Mapping Instructional Technique on High School Geometry Students and Their Ability to Write Geometric Proofs

Geometric proofs provide opportunities for students to practice and demonstrate logical thinking and reasoning, an important skill in any mathematics class. Although proofs play an important role in California's (2006) High School Geometry standards, my 2007-2008 high school geometry students were having difficulty completing proofs of geometric theorems and statements despite many attempts and methods I tried with them. Most of my students knew how to write the steps that included the given information for a proof because we spent a great deal of time discussing the "easiest step." This "easiest step" was simply to copy the given information first and then see what else we knew or could be determined from looking at that given information, but few could write the intermediate steps that should follow from the given information or the other steps that could be identified simply by looking at the picture. Only a very small percentage of the geometry students could write a complete and accurate proof. Many of the students who wrote some intermediate steps wrote steps that were incorrect or unusable and that did not apply at all to the proof they were supposed to be writing. Figure 1 below shows examples of student work, one with incorrect steps and the other with true but unusable steps.

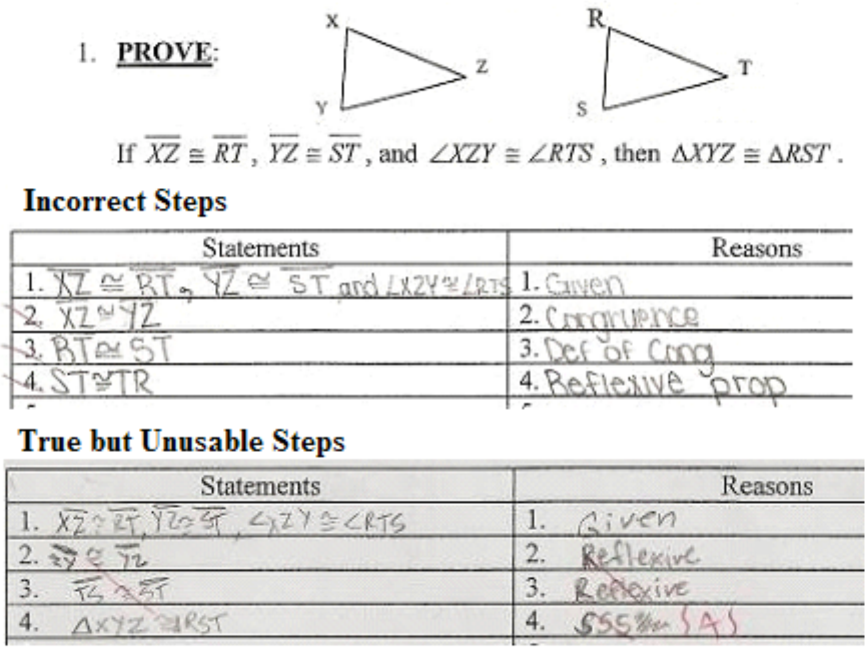


Figure 1. The first work sample shows Omar's[1] use of the definition of congruence to say that the segments RT and ST are congruent which is incorrect and the second is an example of Cindy's use of the reflexive property to say the segments TS and ST are congruent which is true but does not further the completion of the proof.

A majority of my students were able to take the information presented in theorems and apply it to the problems that used the information, but most could not see how the same theorems related to writing a proof or how the theorems and postulates justified a step in the proof and they did not know how to connect one piece of information to another in the proof. The fact that the students seemed to know how to start the proof, even if only out of copying what was said in class, and could use the geometric theorems and postulates to solve other problems, led me to believe that the biggest weakness for the students came in finding a path to get from what they already knew about the picture to what they wanted to show to be true. Weber (2001) states, "The ability to construct proofs

[1] This name and all subsequent names are pseudonyms.

is an important skill for all mathematicians. Despite its importance, students have great difficulty with this task… undergraduates often are aware of and able to apply the facts required to prove a statement but still fail to prove it" (p. 101). This is a similar situation that I found with my own students. Although Weber is referring to college students in the quote above, apparently the problem is not limited to them and could actually be starting much earlier in high school.

In the 1950s, Dina and Pierre Van Hiele developed a learner model with various levels of geometric understanding. In this model, Van Hiele and Van Hiele (1957) claimed that students moved through fixed phases or levels of geometric understanding and that one level must be reached before a student can progress to the next level of understanding. On the Van Hieles' scale, a 5 is the highest and 1 is the lowest level of geometric knowledge of concepts and geometric understanding.[2] This geometric knowledge applied to student knowledge of geometric properties and theorems and their ability to apply the theorems and properties to geometric problems. Table 1 provides a description of each of the levels on the Van Hiele scale.

Table 1

[2] Some writings refer to the Van Hiele levels on a 0 to 4 scale, but for ease of understanding I have changed any use of the 0 to 4 scale to match with the 1-5 scale.

Description of Van Hiele Levels and Student Abilities That Represent Those Levels

Van Heile Level	Description
1- Basic knowledge	Students can identify basic geometric shapes with little or no understanding of the properties relating to them.
2-The Aspect of Geometry	When the student can organize pre-existing geometric knowledge (shapes) into categories like symmetry and learning names and properties of basic shapes(about 20 lessons to achieve)
3-The Essence of Geometry	Still lacking proper vocabulary but can explain important principles like the sum of the measures of angles in a triangle is the same as a straight angle, and understand the concept of congruence and parallelism. (50 total lessons- end of year 1)
4- Insight into the theory of geometry	Can use the concept of parallelism to see the implications that congruent angles are formed, begin to study deductive systems of propositions. Include definitions and propositions.
5-Scientific insight into geometry	Study the system of propositions; study various deductive systems in the field of geometry.

In their writing, Van Hiele and Van Hiele (1957) claimed that only students who reach a level 4 or 5 on their scale of geometric understanding were capable of writing an accurate geometric proof. According to the Van Hieles, students below Level 4 should not be able to do proofs because they have not, “mastered sufficient second level thinking structures” (Van Hiele & Van Hiele, 1957, p. 219). The Van Hiele model used a method for teaching that would allow students to follow a natural progression from one level to the next by scaffolding the information that was presented to the students. This research showed a positive correlation between students’ basic understanding of basic geometric facts and their ability to write an accurate and true geometric proof. Still, while Van Hiele and Van Hiele's research did show a positive correlation between geometric knowledge and proof writing ability, they did not demonstrate a causal relationship between the two. It is important to keep in mind that while it may be necessary to have a basic understanding of geometric properties to write a proof (to reach level 5), it is not a sufficient condition to have a basic knowledge to be able to write a proof, as I discovered is the case for many of my students. Also, it is important to note that Van Hiele and Van Hiele did not claim that the basic knowledge would be enough for a student to reach level 5; the students needed to progress in their thinking about relationships between the concepts in order to reach that level of scientific insight into geometry.

Investigating Van Hiele and Van Hiele's claims about geometric understanding, Senk (1989) suggested that the levels that Van Hiele and Van Hiele posited as requirements for writing a successful proof may have been too high. Her study conducted with 241 geometry students showed that students at lower levels of geometric understanding (level 2 or 3 on the Van Heile scale) could also be successful. In Senk's University of Chicago School Mathematics Project (UCSMP) series of books, she uses a very traditional approach to teaching proofs by stating some given information and then asking what can be concluded from that given information. She continues on in this manner until a complete proof is achieved. Senk's approach utilized teaching the process of proof writing to the students by breaking it down into smaller steps. She achieved greater success with the students on the lower levels of the Van Hieles' scale and their abilities to write proofs. She found that they could be successful at writing proofs, contrary to the Van Hieles' writings. Although she noted that it may be difficult for students at the lower levels of the Van Hiele scale to write proofs, it would still be possible for these students to write a geometric proof successfully. Senk's success with students' abilities to write proofs suggests that if her method could be improved upon, then even more students could be successful at writing geometric proofs.

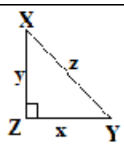
In a survey given the 8th week of school, I asked my class about their least favorite thing in mathematics and 13 out of 34 students explicitly stated proofs and at least 10 others stated something that related to proofs. This is a substantial portion of my class. This self-reported dislike of proofs has led students to react in different ways toward them; some do not even try them on tests; others write only the given information followed by the conclusion; still others add one or two more pieces of information, but very few actually write a complete and rigorous proof. With such limited writing on the students' papers, I had great difficulty understanding where the misunderstandings with proofs were occurring. With the traditional two-column proof, it was very difficult to see what the students were thinking, what they understood, and where breakdowns were occurring in the proof writing process. With such a great dislike for, and difficulty with, proofs it was my belief that it would be very beneficial to see what I could do to change some of the perceptions about these proofs and try to increase students' confidence level with writing them. For this reason, I chose to research this topic further using a method that breaks the proof writing process down into accessible steps and reveals student thinking as they are attempting the proof.

In my investigation into the difficulty of geometric proofs, my goal was to try an alternative method for teaching students how to write a clear and logically accurate geometric proof. In order to do this, I took much of Solow's (2002) suggested method for teaching and doing proofs and modified it for my own intervention. Solow proposes a

"Forward-Backward Method" of writing proofs. While both Solow and Weber's writings center on college level students, many of their writings can be related to high school students as well since I found many similarities with the difficulties my students had with writing proofs and the issues they describe in their writings. Many of the difficulties and processes of writing a mathematical proof are introduced in high school. Therefore, if these processes can be learned earlier in a student's education, they should prevent some of the problems that seem to occur later in college.

Solow's method is about asking a "key" or "strategic" question for how to arrive at the statement we need to prove by starting with the statement instead of the given information and working to write the proof by looking in both directions: starting with the conclusion and working backwards and then looking at the given and working forward, only to go back to the backward method again until a proof is completed (Solow, 2002). Table 2 provides a basic example of how the process would work and some of the description about how the thought processes and steps were determined using this method. Most of the steps presented in the table are to be completed in your head or are the result of a thought process about the proof. They could be written out in a similar manner to the steps presented below and then reordered to form a complete proof that starts with the given information and ends with the statement to be proven but this is not suggested by his writing.

Table 2
Description of an Example from Solow's Forward-Backward Method for Writing Proofs

Step	Description of Process
 Given: Trianlge XYZ has legs of lengths x and y, hypotenuse z, and area $z^2/4$ Prove: Triangle XYZ is isosceles.	This is the proof to be completed.
1) Ask "How can I prove a triangle is isosceles?"	This is the first part of Solow's method that utilizes asking the key question to start the proof. He refers to this as beginning the backward process.
2) Answer the question generally, "show the two legs have the same length or that x = y."	This answer to the key question is general at first and is later adjusted to fit the specific case, i.e. the x = y statement. This is the backward step 1.
3) So I can show x-y = 0 or that $x \le y$ and that $x \ge y$	The backward step 2 can have multiple options. Solow suggests selecting the most likely step so he uses x-y = 0. Since there is no way to continue the backward method, we switched to the forward method. Note that if I can show x-y = 0 then the proof will be complete.
4) I know $\frac{xy}{2} = \frac{z^2}{4}$	This forward step was determined using the formula for the area of a triangle and the given information.
5) I also know $z^2 = x^2 + y^2$	This statement was written using the Pythagorean theorem.
6) I can substitute for z^2 $\frac{xy}{2} = \frac{(x^2 + y^2)}{4}$	This used substitution from the previous two steps
7) Then $x^2 - 2xy + y^2 = 0$	Using algebra we can rewrite the previous equation
8) So, $(x - y)^2 = 0$	Factor the above
9) and, x-y = 0	This completes the proof by matching what we needed from the backward process.

The key question is vital to the success of Solow's method since it is what determines how the proof will be attempted. This method of focusing on what was most important about a proof was different than any that I had tried up until that point and because it is supported by Weber's research and findings, it held promise that it would provide my students with one more opportunity to understand the process of writing a proof and be successful at it. Solow uses a very strategic process that asks the students to think about multiple pieces of information at the same time and find a way to connect them. This made me consider the question, ***"Can Solow's Forward backward method for writing proofs be adapted successfully to meet the needs of high school geometry students?"***

The difficulty for many of my high school students was that they had great difficulty maintaining all of these important pieces of information in their heads at the same time. In response, my intervention attempted to fix this by creating a more concrete procedure so that the students that had not yet developed fluidity at this type of thinking process would also be able to follow the method for writing a proof. I created this concrete procedure by having the students draw the connections that would allow them to exceed their ability to store the information in their short term or working memory and not lose their place because they could simply look at the diagram to see what they had already figured out without losing track of what they had already accomplished. I used Solow's central idea of developing a key question to help maintain the open-endedness for the students and take away some of the anxiety of being wrong because they would be able to see multiple solutions and the process would also include a procedure for eliminating unnecessary information. My hope was that the visual aspects and open-endedness of this new method would help my students become more successful at completing a proof.

One major concern that I also had for my English Language Learners (ELL) and other students with difficulty processing oral language was that in the process of writing a proof there is a great deal of discussion about what step should be written next in the two-column proof but very little information written down during the course of this discussion. The only thing that is actually written down is the end result of the discussion, or the next step in the proof. This extended discussion with so little written information could have created an issue for my ELL students as well as any students that had difficulty processing verbal language. The students that did not have a complete grasp of the English language would have missed out on large portions of the discussion and would therefore not understand how the next step was decided on and added to the proof.

The method I chose to address this problem was the modification of Solow's "Forward/Backward" method of writing proofs using diagrams to visually represent the connections and therefore emphasize more of the discussion points. While I did not have a large number of language learners in my geometry class in the 2007-2008 year, proofs had posed problems for the language learners that I did have which is why I modified the method to try to accommodate them as well. As a result, another big aspect of the visual "proof mapping" was that many of the English-Language Learners in my class were to be given greater access to the process without being put at a disadvantage for not understanding all of the language used to describe the proof writing process. It was expected that this visual organizer could have greatly improved students understanding of what a proof is trying to do even if they still struggled with writing it entirely on their own at first.

Weber (2001) proposes that students need "strategic knowledge" to be able to write a proof and, "Since strategic knowledge is heuristic, designing activities that will lead students to acquire this knowledge will be a formidable task. However,… not putting any effort into such instruction will ensure students will not acquire this knowledge" (Weber, p. 116). A big part of my intervention was about teaching this elusive "strategic knowledge" that Weber defines as "knowledge of how to choose which facts and theorems to apply" (Weber, p. 101). As I tried to help my students to improve their understanding of what goes into a proof, I hoped that they would begin to acquire more knowledge of the definitions and geometric content while learning the process of mathematical demonstration or proofs. So for my inquiry study I planned to try and answer the question "***In what ways does the proof mapping instructional technique affect high school students' performance on writing accurate and logical geometric proofs?"***

Hanna (1971) concluded that students were better able to arrange statements and reasons that were already listed than to tell whether a given statement could be proven with some given information or to answer multiple choice questions with diagrams that asked them what was given and what needed to be proven in the form of a picture. This result has implications for my study and suggested that my students might have similar difficulty writing steps because they were not given a list of steps to put in order and the ultimate goal was for them to start with a diagram or statement and write a proof based on what they saw and that was where Hanna's students struggled. But since the students in his study were better able to put the statements and reasons in order once they were listed, my belief was that this intervention should support this type of writing process as well by having the students find as many statements and reasons as they could before trying to write a proof instead of trying to find the right step, and thus the correct proof, during their first attempt. The intervention should also have helped the students overcome the difficulties that they had with diagrams because part of the intervention had them learning how to break the diagram into the pieces of information it presents and focus on how to use these diagrams to help them write a proof.

Many of my students had very low confidence in their own ability to write proofs prior to the intervention so, as a result, I decided to try to answer the question, ***"In what ways does the proof mapping instructional technique affect students' confidence in their own ability to write proofs?"*** and to see if the confidence might translate to more students being willing to help their classmates with a proof I decided to investigate whether or not students ***"Are more inclined to offer help to other students after they learn the proof map method?"***.

To approach the questions that Senk's research raised about students with less understanding of geometric knowledge having more difficulty writing proofs simply because they were not familiar with the names for the justification for each step, I decided to investigate ***"How does the student's ability to memorize theorems influence student ability to use the proof map method?"***

Finally, to see what the effect proof mapping will have on the students' participation in writing proofs I wanted to find out, ***"To what degree will students attempt a geometric proof if they learn the proof map method compared to leaving that test question blank?"***

I planned to begin this proofs process by using visual diagrams for proof organization which allowed the students to "map out" possible ways to prove the statement that they were trying to prove. This visual model, or "proof map", of the proof should have helped more students see the proof forming rather than having to understand only the language being used. Since Solow's method fits with Weber's and Senk's research conclusions, I chose to use this model for an intervention. So to summarize my inquiry study questions, I want to investigate the research question "In what ways does teaching the "proof maps" affect high school students' performance on writing accurate and logical geometric proofs?" and the sub questions:

1) How does the student's ability to memorize theorems influence student's ability to use the proof map method?

2) To what degree will students attempt a geometric proof if they learn the proof map method compared to leaving that test question blank?

3) In what ways does using the proof map method affects students' confidence in their own ability to write proofs?

4) Are students more inclined to offer help to other students after they learn the proof map method?

5) Can Solow's "Forward/Backward" method for writing proofs be successfully adapted to meet the needs of high school geometry students?

Method

Participants

I chose to investigate the issues of writing geometric proofs by examining my second period 2007-2008 geometry class. This class had 32 students, language learners at almost every level, and one special education student. These students had various academic strengths and weaknesses. This geometry class had California Standardized Test (CST) scores that fall in the basic understanding range with an average score of 3.5-4. This can be seen by the average CST scores in Table 3; a score of three is considered an average score. The specific scores for different CST tests taken by the students can be seen in the charts below. The scores reported in English and math show that although the students in my geometry class scored in the basic, advanced, and proficient ranges for English, there were far fewer proficient students in the math section. So it seemed that there was a gap in their mathematics comprehension that did not exist in other subject areas. These scores also showed that the students were not performing at very high levels academically and generally scored in the average range even if they were performing better in English than in mathematics.

Table 3

Geometry Period CST Scores for 2006-2007 School Year (1 is the lowest and 5 is the highest)

Period 2 Geometry 32 Students Total.

Score	Count	**CST: English Language Arts**	Score	Count	**CST: Math**
1	2	6%	1	2	6%
2	1	3%	2	4	13%
3	10	31%	3	12	38%
4	8	25%	4	9	28%
5	9	28%	5	2	6%
N/A	2	6%	N/A	3	9%

Note: N/A represents students that were not tested on that subject

The students in this geometry class attended Dover High School in Easton, California. At this time, Easton was a growing city in Northern California and had a diverse population. Dover was an older comprehensive high school that had been recently updated and renovated. The campus was large with many tall trees, separate wings for different subject areas, and held a population of about 2,000 ninth to twelfth grade students. The demographic population at Dover High was diverse and included many ethnicities (see Table 4.) The table shows that the demographics in the second period geometry class are close to the demographics of the school as a whole, but there are slightly higher percentages of Filipino, Caucasian, and Asian students and a slightly smaller percentage of Hispanic and African American Students.

Table 4

Ethnicity of High School Students in 2nd Period Geometry Class by Number and Percentage Compared to School Population

Ethnicity	Number of Students (Class) (*n* = 32)	Percentage of Population (Class) (*n* = 32)	Number of students (School) (*n* = 2,004)	Percentage of Population (School) (*n* = 2,004)
American Indian/ Native Alaskan	0	0%	17	0.8%
Pacific Islander	0	0%	30	1.5%
Filipino	2	6.25%	55	2.7%
Hispanic or Latino	4	12.5%	379	18.9%
African American	3	9.38%	284	14.2%
Caucasian (not Hispanic)	20	62.5%	1,098	54.8%
Asian	3	9.38%	122	6.1%
Multiple or no response	0	0%	19	0.9%
Total	32	100%	2,004	99.9%

My second period geometry class had only one beginning language learner. Table 5 shows student language proficiency by number of students as well as the percentages of students that scored at each California English Language Development Test (CELDT) level.

Table 5

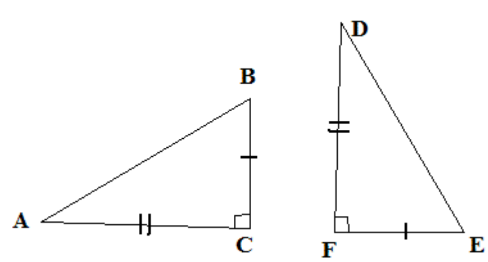
CELDT Scores of 2nd Period Geometry Students by Percentage of Class (1 is the lowest, 5 is the highest)

Score	Count	Percentage
1	0	0%
2	0	0%
3	1	3%
4	3	9%
5	3	9%
N/A	25	79%

Intervention

My intervention plan involved students developing proof maps to help them develop two-column geometric proofs. The technique was adapted from Solow's (2002) "Forward/Backward" method with a few additional modifications to help address my students' needs. For my intervention, the students were asked to begin each proof by examining the statement that was to be proven. If we look at Figure 2 for an example, we can see that the students were asked to prove that two triangles (ABC and DEF) were congruent. Upon introducing the proof, it was very important to emphasize the statement that was to be proven, in this case that triangle ABC is congruent to triangle DEF, which is why I listed that statement before the given information.

Prove: $\triangle ABC \cong \triangle DEF$



Given:

$\overline{AC} \cong \overline{DF}$

$\overline{CB} \cong \overline{FE}$

$m\angle C = m\angle F = 90^o$

Figure 2. Example of one type of proof my students were asked to complete.

Once we had clearly identified exactly what needed to be proven, it was time to formalize a "key question" about that statement. The "key question" is what forced or allowed the students to think more openly about the proof in terms of a general case instead of a specific case. Solow (2002) wrote that the key question should be:

> posed in an abstract way…[because] by asking the abstract question, you call on your general knowledge of triangles, clearing away irrelevant details, thus allowing you to focus on those aspects of the problem that really seem to matter.

The question should have focused on the final step of the proof or the statement that needed to be proven and asked "in general what is my goal for the proof am I trying to write?" so that it could be thought about in multiple ways. It was important to generalize the statement to represent any two triangles, not just the two we were looking at in this example. The key question needed to be written strategically, generally enough to think about all possibilities without eliminating possible routes to write the proof, so as not to close down the multiple ways to approach this proof (Solow, 2002). For the example in Figure 2, a key question for the statement is "how can I prove *any* two triangles are congruent?" This question allowed the students to keep an open mind and focus on all the possibilities for how to approach this proof and thus reduce the possibility of eliminating a method that will easily prove the statement. A proof map provided a visual framework for implementing Solow's "Forward/Backward" process. The given statements were enclosed in circles, and the arrows pointing to the circle represent the reason that the circles were there. The circles became the "result" of the given. The conclusion was also included as a circle. At the early stage of the proof map process, it was not yet clear what possible reasons could generate the conclusion. Solow's key question was entered as "thought cloud" as an aid to help students think about possible reasons for the conclusion. Figure 3 shows the beginnings of how to set up a proof map before we have answered the key question.

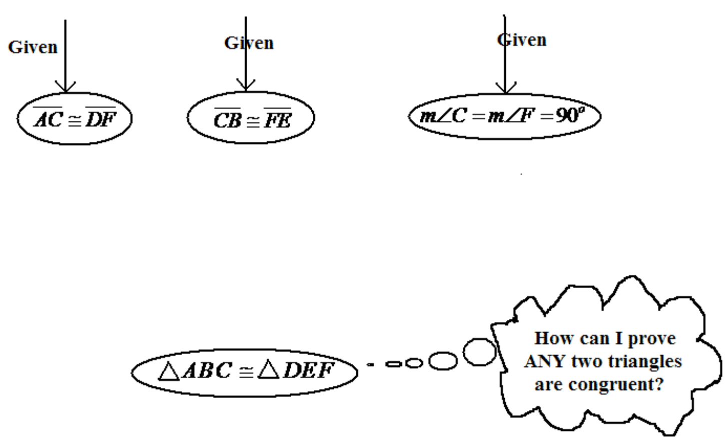


Figure 3. Example of the proof map students could set up for the geometric proof example in Figure 2

Once the key question has been asked and entered into the thought cloud, students attempted to answer it. Continuing with the example in Figure 3, my students had learned six different postulates and theorems that would allow them to prove two triangles were congruent. This information was entered into the proof map by drawing six arrows, each representing a reason, that point to the conclusion circle, as shown in Figure 4. In the backward step 1, the student was to list all of the possible ways to prove any two triangles were congruent and then in the forward step one they listed any information that could be derived from the given information as well as the part of the triangle it represents (side or angle) and any other information about the triangles. Any information that can be justified by a reason is circled and any information that cannot be justified by a reason is left un-circled until its validity can be determined. The key question is represented by a thought bubble off to the side of the statement to be proven and the reasons for each step will be represented by a directional arrow with the reason pointing to the step. This allowed easy transitions to the two column proof format later. The lines connected the valid conclusions from the given information and any data that could not be concluded from the given information or otherwise justified by a postulate of theorem was not circled.

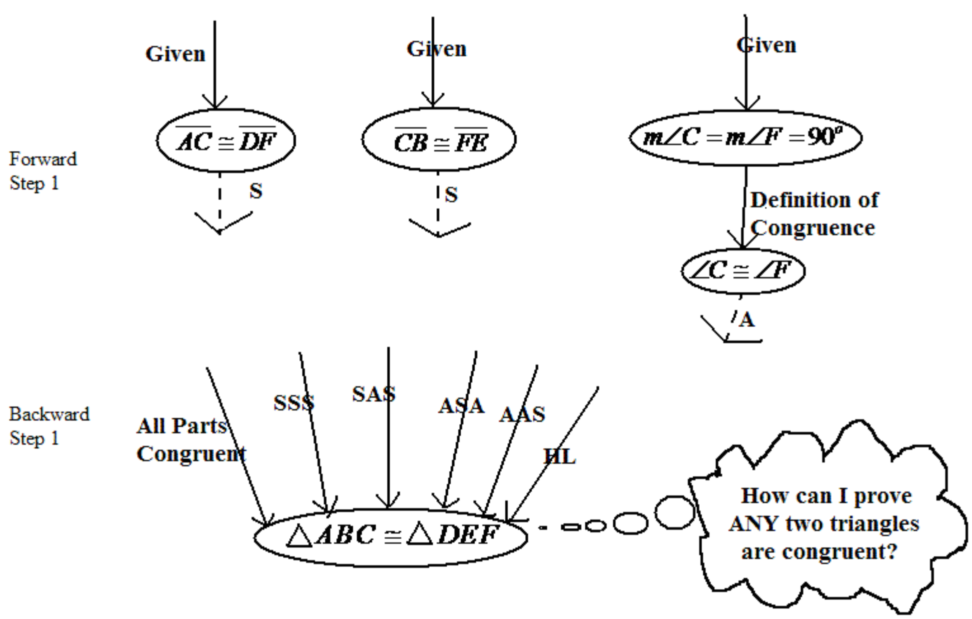


Figure 4. Backward and forward step 1. The dotted line represents a note to the creator that the “S” pieces of information represent a side of the triangle and the “A” pieces represent an angle so he/she can better connect the information and complete the proof.

This proof map, which functions as a visual organizer, was not suggested by Solow, but in order to provide more access for my language learners I developed a visual model that employed all of Solow's principles but illustrated them in a graphical map. One other change from Solow's model is that I started this process by listing all of the ways to prove two triangles are congruent, whereas Solow says to pick the most strategic way by looking back at the given information. I chose to introduce this using all the ways the first few times and gradually we discussed ways to eliminate and choose the best method by crossing out the path and giving a reason why that is not a practical way to write a proof, but only after they had learned the basic method first.

Once the first steps were completed (both backward and forward) the students needed to find a way to connect the two key parts, the given and the statement to be proven. This was accomplished by working backwards again. In this example the students examined all the ways we can prove two triangles are congruent and see what pieces they needed in order to use each theorem or postulate. In this example the Side-Angle-Side congruence postulate would use three pieces of information that could be derived from the given information. We can see the connections in Figure 5.

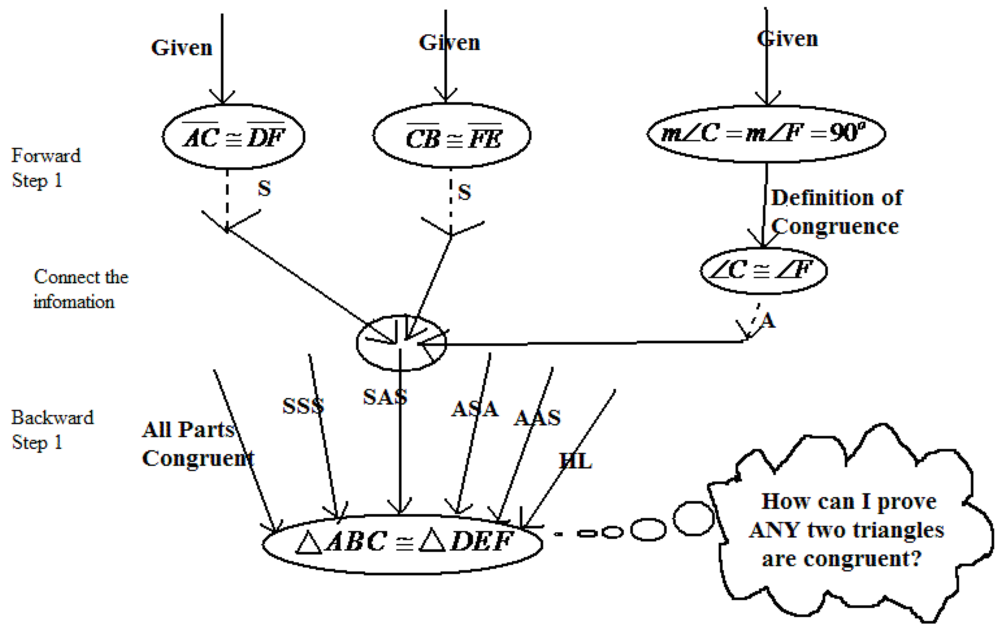


Figure 5. Step 3 in the mapping process that connects the backward and forward steps to complete the proof.

Once a competed pathway was found, we eliminated all other information on the chart and the only thing left to do was to organize the path into a formal proof. I chose to use two column proofs because most of the California standardized tests used this format so I wanted the students to be familiar and comfortable with the two column proof's design and function. Figure 6 shows the completed path with all extraneous information crossed out.

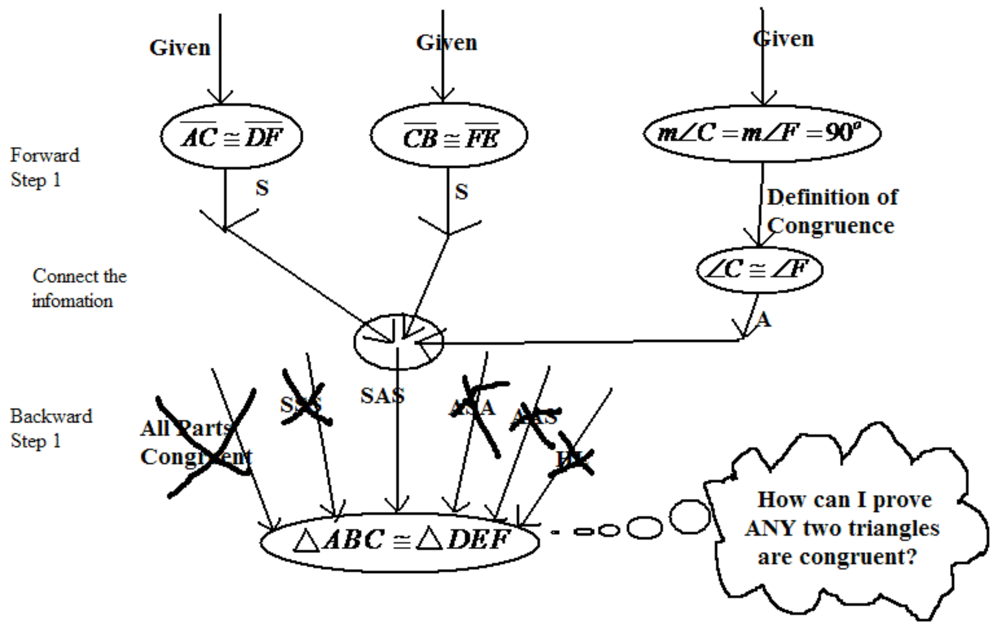


Figure 6. Completed proof map that connects the given to the statement to be proven.

The formal proof was written after the reasons have been sorted out using the organization model and the example for this proof can be seen in Figure 7.

Steps	Reasons
1) $\overline{AC} \cong \overline{DF}$	1) Given
2) $\overline{CB} \cong \overline{FE}$	2) Given
3) $m\angle C = m\angle F = 90^o$	3) Given
4) $\angle C \cong \angle F$	4) Def. $\cong$
5) $\Delta ABC \cong \Delta DEF$	5) SAS

Figure 7. Completed two column proof translation of the completed proof map.

This method was different from anything else I had tried with my students up to that point and I hoped it would help with their thinking about how to write proofs and how to understand proofs that have already been written.

Data Collection Procedures

I began collecting data by administering a standard treatment posttest to my second period geometry class on November 28, 2007. This test consisted entirely of geometric proofs that they had already been instructed on writing using the standard treatment of starting with the given information ant then seeing what could be concluded from there (see Appendix A.) I looked at each test question to categorize the students responses based on a 0 to 5 point rubric where 0 indicated a blank problem and a 5 was a completely accurate proof. Once I collected the achievement data, I had the students take a survey of both Likert scale rating questions and free-response questions to investigate their feelings about proofs before we have started the intervention (see Appendix B.)

On November 30th, 2007 I began the intervention by introducing the "Forward/Backward" proof mapping method to the students. We began using cloze-like proofs that had a two-column proof written out for the students with some statements missing and some reasons missing. This allowed the students to first practice just setting up a "proof map" and work on finding the connections between ideas before they began writing the maps from scratch. They then worked through three proofs as a class using this mapping method. During this introduction, I emphasized the importance of identifying the key question and how we can answer it. Once we had done this together as a class, I asked the students to work on two more proofs using this method in small groups of three to four students. While the students were working, I monitored their involvement and interest in the new method by doing sweeps of the classroom and counting the number of students that are on task and the number that are helping other students. Using this technique over the course of the intervention allowed me to see how many students stayed involved and after how long they lost interest as well as how many students were willing to help their classmates.

On December 7th, 2007 for about 30 minutes I went over two more proofs with the students using the proof-mapping method. This time I asked the students to tell me how to fill out the map with more frequency than they did the first time, which meant that I asked the students to tell me the next steps more often and did not guide them through the proof mapping process as much. While we were doing this, I monitored student involvement using sweeps and counted involved students. Then again on January 8th, 2008 (after the 2.5 week winter break) for another 30 minutes the students and I went over two more proof maps as a class before I asked them to work in groups to complete two on their own. The first one required the students to construct a map from a partially completed, two-column proof and the second asked them to write one from scratch. Two to three times weekly from January 9th, 2008 to January 18th, 2008 for about 30 minutes each, the students and I discussed ways to strategically write proofs by looking at both the given information and what needs to be proven. I again monitored student involvement.

On January 22nd, 2008 I administered my closed note quiz on proof mapping, that allows the students the use of a list of theorems and postulates, to get a basis for comparing to the quiz, which was followed on January 29th, 2008 with open note quiz that allowed the students to use the list of postulates and theorems, and which allowed me to compare to the closed note quiz (see Appendix D for complete copies of each quiz.)

My final data collection occurred on February 8th, 2008 when I administered my final posttest to see if there was any improvement in students' ability to write geometric proofs using this new method. I compared the initial results from the standard treatment posttest and the final results of this test to see if there was improvement. At this time I also administered a post-survey to see if students' perception of their own ability and confidence had changed either. At this point I analyzed all of the data and looked for trends in the data to see how this method affected students' ability to write geometric proofs.

Results

In order to answer the research question regarding student achievement using "proof mapping," I used a posttest-posttest comparison of student achievement. To begin the data collection, I had the students in my second period geometry class take the standard treatment posttest that asked them to write proofs about triangle congruence, a topic we had already gone over in class using the standard method for teaching students how to write proofs (see Appendix A for a complete copy of the test.) I used the results from this standard treatment posttest to check for improvement on the proof mapping treatment posttest. Each proof was scored from 0 to 5 points using a rubric (see Table 6).

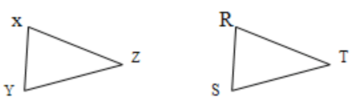
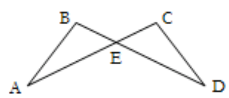
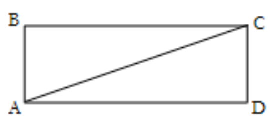
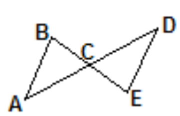
Table 6
Rubric Levels and Description

Score	Description
0	Question left blank
1	Given information with no additional steps
2	Given information with one or two correct additional steps
3	Student has important information included in the proof but is lacking a key step to complete the proof
4	Student made one or two minor errors
5	Correct Proof

In order to reduce bias in the scoring process, I first placed student pseudonyms on each test and removed or covered their true names entirely. I scored the tests two days later to ensure that any memorizing of the pseudonyms was minimized.

Once the proofs were scored, I calculated the average score on each posttest proof as well as the standard deviation and t-value for each. The proofs on each posttest were substantially different in difficulty and method, so I analyzed each proof individually. The scores can be seen in Table 7.

Table 7
Mean Student Scores per Question on Standard Treatment Posttest and Proof Mapping Posttest with Standard Deviation and T-Values

	Standard Treatment Posttest (n = 32)		*Proof Mapping Treatment Posttest (n = 32)*		
Test question	*M*	*SD*	*M*	*SD*	*t*
 If $\overline{XZ} \cong \overline{RT}$, $\overline{YZ} \cong \overline{ST}$, and $\angle XZY \cong \angle RTS$, then $\Delta XYZ \cong \Delta RST$.	2.60	1.32	3.75	1.62	4.36***
 If $\overline{BE} \cong \overline{CE}$ and $\overline{AE} \cong \overline{DE}$, then $\Delta ABE \cong \Delta DCE$.	2.31	1.39	2.75	1.67	2.30*
 If $\overline{BC} \cong \overline{DA}$ and $\overline{BC} \parallel \overline{AD}$, then $\Delta ABC \cong \Delta CDA$.	1.97	1.00	2.59	1.48	2.86**
 If C is the midpoint of $\overline{AD}$ and $\angle A \cong \angle D$, then C is the midpoint of $\overline{BE}$	1.44	0.98	2.23	1.50	4.25***

*p<.05, two-tailed. **p<.01, two tailed. ***p<.001, two-tailed.

As can be seen in the above table, the students' average scores on each proof on the standard treatment posttest show that, in general, they could do little more than write the given information and one or two steps that did not necessarily help the proof, which demonstrated a very low understanding of how to write a proof by the students. The average scores on problems 2, 3 and 4 were 2.31, 1.97, and 1.44 respectively. In contrast, the proof mapping posttest showed that most students were able to form at least a general idea of where the proof should look like as demonstrated by the average scores of 3.75, 2.75, and 2.59 on questions 1, 2, and 3 respectively. I set the acceptable probability that there was a change in the data by pure random chance at 0.05 for each item, which was statistically analyzed in this data and any other data that follows. This level of accuracy ensured that no more than 1 in 20 items would have changed purely by chance. There was a statistically significant increase in the scores after the intervention on all four proofs. And two of the four proofs had a probability less than .001 and one of the four had a probability of less than .01, so they far exceeded the minimum requirement for statistical significance. For this particular set of data I also set 0.5 point improvement as having practical importance from my teacher's perspective. I decided that an overall average improvement on a problem as a class of 0.5 points meant that the class improved by 10% on that particular proof and that would represent an entire letter grade as a class, which is a large improvement for a class. Every problem, except problem 2, had their average score increase by at least 0.5 while problem 2 increased by 0.44 points. Figure 8 shows a comparison of students' scores before and after the intervention.

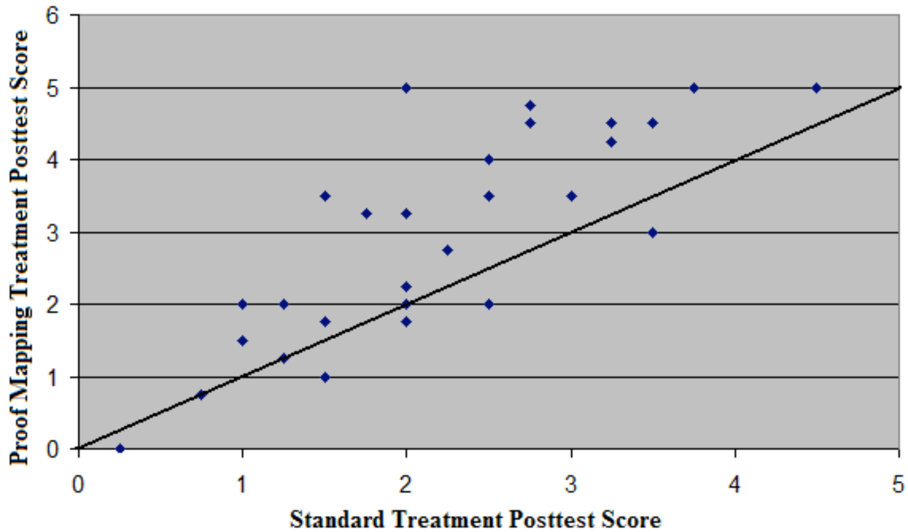


Figure 8. Comparison of average standard treatment posttest scores per student across the whole test to average proof mapping treatment posttest scores by student. The black line represents no change in the scores so all of the points above the line represent an increased score while the point below shows a drop in their overall score.

Figure 8 clearly shows that there was substantial improvement in scores after the intervention. And of the students that decreased their scores, not one fell by more that one half of a point.

To answer my first sub-question, "How does the student's ability to memorize theorems influence the students' ability to use the proof map method?" I compared students' results from a test that allowed them to use a list of postulates and theorems to a test that did not allow them to use this list. The tests were administered one week apart and I wrote two distinct tests that had proofs that required similar steps to complete. The tests each had 3 questions and the 3^{rd} question was the same for both. To encourage a fair test I left one week between these two examinations to prevent the students from remembering too much about what they wrote on the first test and administered the second test before I returned the first one to them so they did not have an opportunity to make corrections before the second test. When I compared the scores on each test there was little difference in the mean scores for the test. Some students, about 10%, wrote slightly more information on the quiz the second time when they had the list of theorems but about half of these students wrote incorrect or unusable steps simply because they found a theorem or postulate that they liked and tried to make it fit even if it really did not apply. Figure 9 shows an example of Michael's test that used the list of theorems and postulates. He used the perpendicular bisector theorem even though there was no mention of a bisector in the problem.

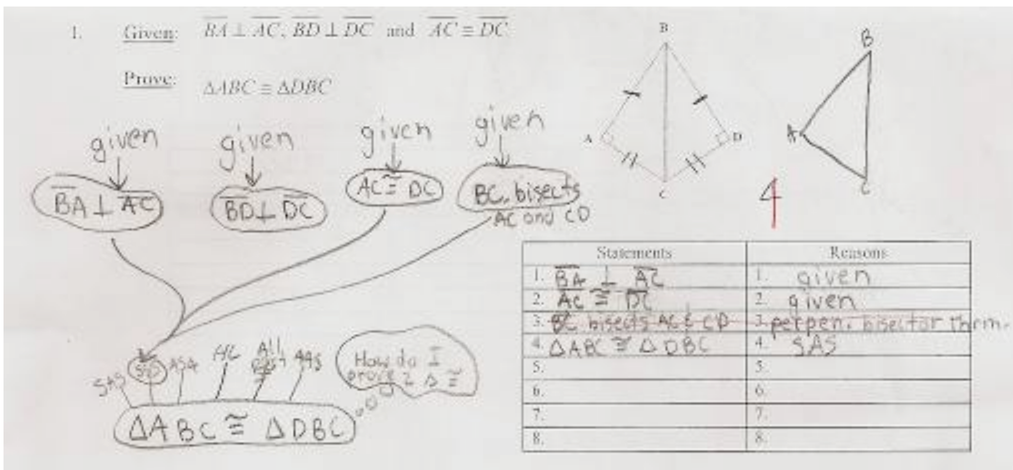
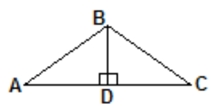
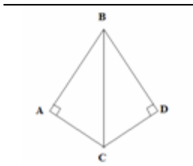
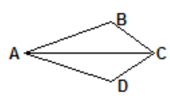
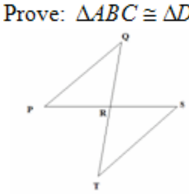
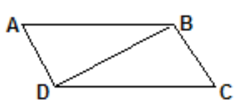
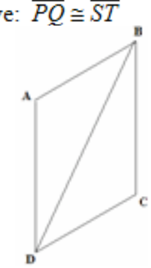


Figure 9. This is a sample of Michael's work on the test using the list of theorems and postulates. Note the use of the Perpendicular bisectors theorem to justify the unnecessary statement in step 3 of the two-column proof.

For some students, this list of theorems meant that they moved away from what they knew for sure and wrote less accurate proofs when they had the list. A comparison of the third problem for both tests demonstrates the actual effect that this list of theorems had on the students. Table 8 shows the results of the scores on the tests with and without the list of theorems. I focused on question 3 for each because it was the same proof and in total 13 students changed their scores on problem 3 and 3 of these students had a change of 2 or more points. That is a total increase of 16 points on this question as a class.

Table 8

Student Scores on Quiz with and without a List of Theorems and Postulates

Test Questions without Theorems	Test Questions with Theorems	Without Theorems (n = 33) *M*	Without Theorems (n = 33) *SD*	With Theorems (n = 33) *M*	With Theorems (n = 33) *SD*	*t*
 Given: $\overline{BD} \perp \overline{AC}$ and $\overline{BA} \cong \overline{BC}$ Prove: $\Delta ABD \cong \Delta CBD$	 Given: $\overline{BA} \perp \overline{AC}$, $\overline{BD} \perp \overline{DC}$, and $\overline{AC} \cong \overline{DC}$ Prove: $\Delta ABC \cong \Delta DBC$	1.91	1.04	2.30	1.59	1.81
 Given: $\angle BAC \cong \angle DAC$ and $\angle BCA \cong \angle DCA$ Prove: $\angle B \cong \angle D$	 Given: $\overline{PR} \cong \overline{SR}$, and $\overline{QR} \cong \overline{TR}$ Prove: $\overline{PQ} \cong \overline{ST}$	3.21	1.71	2.21	1.62	-4.42***
 Given: $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$ Prove: $\angle A \cong \angle C$	 Given: $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$ Prove: $\angle A \cong \angle C$	1.79	1.39	2.18	1.47	2.74*

*p<.05, two-tailed. **p<.01, two tailed. ***p<.001, two-tailed.

In analyzing the data found in Table 8, there were two things that stood out. First, question number 2 was not a good comparison question. The average score on question 2 decreased by a full point (and statistically this was not an accident) so I further analyzed the questions and realized that, while the first proof was an Angle-Side-Angle postulate proofs using the reflexive property, and the second proof was a Side-Angle-Side postulate proof using vertical angle theorem, and the number of steps is the same and they seem comparable, in reality the students struggled a great deal more when they had to use the vertical angles theorem than when they needed to use the reflexive property of congruence. As the second test actually created a more difficult problem, I did not use this question to help in answering the research question. The second item that stood out was question 3. The average score increased from 1.79 to 2.18 on the same question (with a slightly different orientation of the picture). This increase was found to be statistically significant using the t-test. This result indicated that students did perform slightly better when they had the opportunity to use the list of theorems and postulates, even if the average scores still meant that they could identify information in the picture, but still did not know how to write the rest of the proof. Still, because of the nature of the incomparability of problems 1 and 2, the results from problem 3 do not say much about the effects of memory on the ability to write a proof. There is no consistent pattern or trend and so these results are inconclusive at this point.

To answer my second sub-question, "To what degree will students attempt a geometric proof if they learn the proof map method compared to leaving that test question blank?" it was a simple matter of comparing quantities of written work in students' pretests and posttests to see if more correct information was included. Table 9 shows the counts of the number of steps included in the two-column proof on each posttest. Even incorrect steps were included in the counts.

Table 9
Mean Number of Steps Written on the Two-column Proof for each Posttest.

Proof Number	Mean number of steps on 1st posttest (n = 33)	Mean Number of steps on 2nd posttest (n = 33)	Change
1	3.81	4.06	0.25
2	3.53	4.17	0.64
3	3.75	4.05	0.30
4	4.53	4.09	-0.44

This table does not seem to demonstrate a great change in the amount that students will attempt a proof, based on the number of steps alone. However, for many students the number of steps is irrelevant because they have good quality steps, so in order to eliminate counting steps of students that "knew what they were doing" I reanalyzed the data by removing any students that averaged a 3.0 or higher on either posttest because these students were not the focus for this question. They were not having difficulty starting the proofs and they were writing high quality steps, so the quantity was not a factor for them. I looked at the numbers again for the students that were having difficulty (see Table 10.)

Table 10
Mean Number of Steps Written on the Two-column Proof for each Posttest excluding students with 3.0 average or above on either Posttest

Proof Number	Mean number of steps on 1st posttest (n = 17)	Mean Number of steps on 2nd posttest (n = 17)	Change
1	3.59	4.18	0.59
2	3.18	4.35	1.17
3	3.06	4.24	1.18
4	3.71	4.47	0.76

When the higher performing students are taken out of the equation, there is a much higher increase in students writing steps and attempting the problems. These additional steps added 0.32 points increase in the overall average on the second posttest compared to the first posttest. So while the initial results from Table 8 seem to indicate that proof mapping may not increase students' attempts at proofs, further analysis seems to indicate that the students that initially had difficulty writing the proofs were coming up with more and more useful steps through the proof mapping technique.

My third sub-question, "In what ways does using the proof map method affect students' confidence in their own ability to write proofs?" needed to be answered by looking at the students' opinions and thoughts. There were two major sources for this confidence data. First, I compared changes in the students' responses to the survey that they completed before the intervention as well as after. The results can be seen in Table 11.

Table 11

Mean student scores on pre and post surveys with standard deviation and t-values

Survey Statement	Pre-Intervention Survey (n = 32)		Post-Intervention Survey (n = 32)		
	M	*SD*	*M*	*SD*	*t*
1) I like writing geometric proofs.	1.38	0.75	1.91	0.89	3.57**
2) I think writing a proof is easy.	1.59	0.80	2.16	1.05	2.81**
3) I like writing proofs because they are challenging.	1.69	1.00	2.03	0.93	1.58
4) I believe that everyone can be good at proofs if they try hard.	3.47	0.95	3.47	0.92	0
5) I think geometric proofs help me think about geometry in different ways.	2.56	1.01	2.72	0.96	0.64
6) I know I can be successful in math if I want to be.	4.00	0.67	4.09	0.96	0.57
7) I would like to learn more about how to write geometric proofs.	3.71	1.18	3.56	1.13	-0.61
8) I think it would be useful to see different ways of doing the geometric proofs.	3.43	1.16	3.16	0.88	-1.22
9) I am confident in my own ability to write geometric proofs.	2.18	0.97	2.75	0.88	4.19***
10) I would feel confident enough to teach another student how to do geometric proofs.	1.44	0.80	1.78	0.97	1.73

*p<.05, two-tailed. **p<.01, two tailed. ***p<.001, two-tailed.

Three of the statements returned a statistically significant change in opinion. Statements 1, 2, and 9 demonstrated an increase in the score that was significant. Statement 9 dealt specifically with student confidence and increased by 0.57 points. The post-survey also had a smaller standard deviation which indicated more agreement in the class that they were more confident in their ability by the end of the intervention.

The second major data source to answer this sub-question was found in the free-response survey questions I asked the students. I focused on the four major questions whose responses dealt with proofs. Table 12 shows the students responses to questions 11 through 14 on the survey.

Table 12
Number of Students that Listed Proofs Explicitly in the Free-Response Portion of the Surveys

Free Response Survey Question	Number that Listed "proofs" Explicitly on pre-survey ($n = 32$)	Number that Listed "proofs" Explicitly on post-survey ($n = 32$)	Change in number from pre-survey to Post-survey
12) What is your least favorite thing about mathematics? Why?	17	19	+2
14) What is the thing you need to work on most in math classes?	20	15	-5
11) What is your favorite thing about mathematics? Why?	0	1	+1
13) When you think about math classes, what is your biggest strength? Why?	0	1	+1

The table indicates that two more students dislike proofs more than they did before starting the intervention but five fewer students felt that it is the thing they need to work on most in math. One of the students who said she disliked proofs the most, Cindy, even stated “proofs maps” not just proofs. Jane also added to her comment that “I would say though that proofs have become a little more clear to me,” despite the fact that they are still her least favorite thing about mathematics. Jane’s comment seems to be true for many students even though they did not explicitly say that. The overall score increase of the proofs test shows that, for many students, proofs were becoming “more clear” even if they were more disliked.

There were 15 students who listed proofs as the thing they need to work on most in math class. This represents a 16% decrease in the class as a whole and 25% of the students that originally listed proofs as their biggest weakness. There is a small decline in the number who felt that they really need to work on proofs even if more students do not like writing them. Stacy added to her comment, “Proof maps, its [they are] hard but I guess I can do it [them].” So while she still thinks she should work on it she also recognizes an ability to be successful using them. Peter wrote, “I still say prove [proofs] a little bit but I’m get [getting] better on [at] it.” Peter also acknowledged a growing strength in his ability to write proofs. Two students even went so far to change their opinions— when Lilly claimed that proofs was her biggest strength in math class (she averaged a 4.5 on the first posttest and a 5 on the second posttest) and Maya stated that her favorite thing about math is, “proof maps that I can understand. I no longer find it as challenging to write a proof since learning how to use a proof map.” Maya’s comment demonstrated a gratefulness to have a tool that helps overcome such a difficult task. All of these student comments indicate that the early exposure to proofs using the standard treatment left them feeling very discouraged about proofs even after they had much greater success after the intervention took place. It seems that the one negative experience stayed with them even after they had been successful with it later.

One other major thing that stood out from the free response survey questions is that the students were better able to identify where they were confused as well. Previously students said the difficult parts were “finding the next step” or “the reasons” but on the post survey many students added some specificity and wrote “trying to remember the theorem or postulate that explains what you’re trying to say” or “the reasons, remembering the theorems and postulates is hard.” These comments reflect a greater comfort with the geometric terminology as well as a better self-awareness of the things in proofs that are difficult for them. Maya commented that the most difficult thing about proofs for her was, “I think the most difficult thing now is making sure I have enough room to do a two column proof after I do the map.” But Billy still felt that he did not like proofs at all and he let me know when he wrote that the most difficult thing about proofs was “Everything, we shouldn’t have to do that crap.” But there were far fewer negative statements like this on the post-survey and this seems to reflect a changed attitude as well as ability.

My final sub-question, “Are students more inclined to offer help to other students after they learn the proof map method?” needed careful data collection to answer. In my baseline observation using sweeps, I watched for one key thing while the students were engaged in activities and instruction involving proofs: which students were assisting another student with proofs. There was a very small percentage of students helping others (approximately less than 1%) before the intervention started. This lack of helpers could have been due to the students’ lack of confidence in their ability to do proofs or just a lack of interest in the success of their classmates. In the preliminary sweeps data collection, I found that at any given time during the activity relating to proofs and with a class of 34 present that day, there was (on average) less than one student helping another in the course of a 40 minute activity. After the proof mapping lesson was taught and the students began an activity using the maps, now with 35 present in class, the number of student helpers increased to at least one per sweep in an hour long activity. By the final proof mapping activity before the completion of the intervention, the students were asked to make corrections to a proof they had already completed and that had been scored by the teacher. The number of student helpers increased to at least three per sweep and many more students were asking the teacher for additional help as well. This does not seem like a dramatic change but, considering that only one student agreed that they would feel confident to teach another student how to write a proof before the intervention and after the intervention, one agreed with this statement, and one strongly agreed, and the average score on this statement changed from 1.44 to 1.78, the 7% to 10% increase is a substantial result for a teacher who is trying to encourage students to work together in a class.

Discussion

In the United States, the dominant technique for recording proofs in high school geometry is a two-column format in which mathematical statements are numbered and listed in the left column and the corresponding reason for each step is given in the right column. Typical instruction focuses on getting the steps listed in the right order with the correct reason. In this study, the two-column proof is viewed as an 'end' rather than a 'means' for producing a successful argument. Proof mapping, a visual adaptation of Solow's (2002) Forward-Backward proof writing technique, was created to allow students to record their provisional thinking about making their geometric argument. After explicitly writing possible steps and reasons in the proof map, students were able to evaluate productive and nonproductive pathways to proving the result. Once a successful path was mapped, the students created a standard two-column proof that reflected the thinking contained in their maps.

Since proof mapping is a new technique, it requires an investigation into all of the potential benefits that it offers and drawbacks that need consideration. Thus, the main research question that guided this inquiry was *In what ways does teaching 'proof maps' affect high school students' performance on writing accurate and logical geometric proofs?* There are two major aspects that need to be discussed when evaluating the effectiveness and utility of the proof mapping technique. The discussion includes an investigation into the effects on achievement and memory. Three major achievement areas were impacted by the proof mapping technique: efforts to complete proofs, connections between statements, and the accuracy of the final proofs.

Efforts to Complete Proofs

One measure of performance in a geometry class is students' willingness to attempt a proof. As noted earlier, prior to the intervention students sometimes left proofs blank, suggesting that they saw no point in even writing one step of the argument, most likely because they did not know how to begin. The number of steps included in a student's proof is an indirect, rough measure of the student's understanding of the argument. For example, a blank response may indicate no understanding of how to begin; providing just the "given" indicates that students may know how to begin the problem; and providing two or three steps may indicate that students know how to make a partial argument with justifications. It is important to note, of course, that number of steps is only a gross measure of understanding. It is certainly the case that the students may add extraneous or incorrect steps to their argument; nevertheless, the number of steps students offer in completing a proof can be though of as a "first cut" at gauging students' achievement in proof writing. To answer the research question *To what degree will students attempt a geometric proof if they learn the proof map method compared to leaving that test question blank?* I found that since the posttests only had questions about proofs, unlike the previous chapter tests that had other questions as well, few students actually left the questions entirely blank. Since this was the case, the research question about whether the students attempted the proof at all was generalized to examine differences in the number of steps on the first posttest versus the second posttest. The initial results for the average number of steps used on each proof showed that there was very little increase on questions one through three and there was even a decrease of the average number of steps for the fourth proof on the second posttest. This showed that as an entire class there was no statistically significant change in the number of steps that students wrote after the proof mapping intervention. Overall, it appeared that the proof mapping method did not change the number of steps that students were willing to present in an attempt at completing a two-column proof.

In an effort to examine the data further and to thoroughly examine the potential effects, I examined a subset of the class and looked at the students that really needed help to get started on a proof. To do this I removed the students who scored an overall average of three or higher on either posttest. I did this because the students that scored a three or higher showed that they seemed to understand how to start a proof and how to find important information from the figures and so they really did not need the intervention to help them in their attempts at the proof, nor were they the focus of the research question as I was trying to investigate the effects on the students with difficulties starting proofs. The students with scores that averaged less than a three were the students who were struggling with the basic aspects of the proofs and, as such, warranted some extra examination. These students ($n = 17$) increased the average number of steps per problem by about one full step on all four problems. In fact, three of the four proofs showed a statistically significant increase in the number of steps. So this subset of students did actually attempt the problems more frequently than they had before the intervention. These extra steps could have been due to a better understanding of what the proof should look like, a stronger ability to recognize information in the diagrams, or simply an extraneous step that they decided to include. Since there was no filtering of whether the steps were germane to the proof and only the number of steps that they used in the two-column proof was counted, many factors could have influenced the number of steps they used. Since this was the case, I checked to see if the extra steps were beneficial to the proofs or just arbitrary steps.

The students that scored an average that was less than three on the posttests not only increased the number of steps by about one per proof, but they also increased their average score by 0.34 points. This indicated that the steps were useful more often than not. So while some students many have written more and not enhanced their proof, a higher percentage did improve the proof and make it more accurate as well. I also found that the seven students that had a decrease in the total number of steps across all four proofs had an increase in score by an average of 0.54 points. This indicated that the steps were becoming more efficient and important for the proofs for these students. If the students with difficulty starting a proof can write more useful steps using the proof mapping method and the students who wrote fewer steps can get more benefit for the steps they chose to use, it seems that the proof mapping technique really has a positive effect on both the usefulness and the quality of the steps used.

Connections Between Statements

A student's ability to recognize the relationships between geometric figures and their properties is an essential ability if a student is to be successful in a geometry class and certainly to be successful at writing a proof. Proofs are a strong tool for measuring the students' understanding of how their own arguments fare in relationship to geometric theorems and postulates but also how the students understand the relationships between diagrams and the theorems that relate to them. A drawback with the standard method for teaching and writing geometric proofs is that there is no way for the teacher to “see” the students’ knowledge of the connections and relationships between the picture and the proof nor for the students to “see” the relationships as they are being presented in the formal two-column proof. The mapping method was developed as a tool to show both the students and the teacher the relationships and connections between the statements that are being presented by the proof writer. For students to be successful proof writers they need to be able to understand the relationships and thus connections between what they know and what they want to be able to say with that information. The mapping method seemed to provide a bridge for many students to begin to make the connections in their understanding.

Prior to the proof mapping intervention, I had few methods for understanding why or how the students were having difficulty with writing proofs. I was unable to determine what the students lacked in their abilities to write proofs because the standard method for teaching students to write proofs required them to only write down the next logical step, presumably after a long and in-depth, though internal, thought process. During one of the first activities using the proof mapping technique, I assigned the students to complete a proof map from a two column proof that was missing a few statements and reasons. Many of the students just filled in bubbles with the listed statements from the two-column proof and then connected the bubbles using arrows sequentially but missed the real connections between the statements and reasons within the proof (see Appendix C.) This sequential ordering of the bubbles pointed out the emphasis on ordering information that is present in the standard method for writing a two-column proof. When the students were first taught the standard method for writing a proof, they were taught to number each step as the proof progressed. The fact that this later translated to sequential ordering of bubbles on the proof map seems to indicate that students' perceptions of a proof is that the process of writing a proof is sequential and can only be written in one way and in one order— like many of the other topics they learn in mathematics. This numerical connecting of ideas showed both a need that the students had to understand that the connections in a proof are semantic rather than syntactic, and the need for the teacher to find a way to demonstrate this difference and emphasize that the numbers are just a way to organize the data after the thought processes were finished. As such, we discussed the aspects of the theorems that allowed us to justify a step in a proof as well as the theorems that required more than one piece of information in order to use it.

The mapping technique provided me with a visual diagram of the students understanding as well as a discussion opportunity for the students and me to talk about when and how the steps are connected. The mapping method emphasizes the importance of keeping in mind the direction the proof should be moving and provides an opportunity to "see" students' thinking processes drawn out on paper. This also translated to higher proof mapping posttest scores as the students were better able to organize their thoughts and structure the proof in a more open-ended fashion which resulted in greater access to the process as well. Once the students learned to "show" their thinking using the proof mapping technique, I was better able to see the errors and explain the strategic thought that is employed in writing a proof which helped the students to develop a better understanding of the true connections between the steps within a proof.

One might say that of course the proof scores would increase after spending two months teaching the proofs and going over the processes, but these students had spent two months working on proofs prior to the intervention and most of them were just as confused when the proofs chapters ended as they were when we began learning how to do proofs, which is evident from the first posttest scores. The test results' increase also seemed too great to have simply been a result of extra practice. Practicing these connections also led to another area of growth for the students— the accuracy of the proofs they were writing.

Accuracy

The accuracy of a geometric proof is perhaps the most sought after result for many geometry teachers. Accuracy is what shows whether or not the students have "put all the pieces together" and understand the process. The measurement of accuracy of a proof is simply a matter of checking for correct steps that are written in a logical sequence and that have the proper justifications. The data showed just how much of an effect proof mapping had on students ability to write an accurate proof.

The increase of the accuracy of the students is best represented by looking at the change in the scores on each question from the first posttest to the second. Each of the four posttest questions showed a statistically significant increase in score. This increase also carried a practical importance as each question increased by at least 0.44 points on average and three of the four questions increased by more than 0.5 points on average. This increase in score showed how much more accurate the students had become in writing a geometric proof. This 0.5 increase meant that the students were able to derive more information from the diagram and make more accurate connections between ideas in the proofs. Accuracy is a very important aspect of any mathematics class and so the more accurate proofs are a highly desirable effect of the proof mapping technique.

As the students became more accurate at writing proofs and saw the increase in their scores, they became more confident in their ability to write proofs. This change in confidence level provided an answer to the research question "in what ways does the proof mapping method affect students' confidence in their own ability to write proofs?" The most telling data sources to analyze this question were questions 1, 2, and 9 from the post-survey. These three statements all showed statistical significance and reflected on students' perceptions and feelings about proofs. The first statement on the pre-survey, "I like writing proofs," had a mean score of 1.38 out of 5 possible (5 being the highest) and on the post-survey this changed to a mean score of 1.91 and the second statement, "I think writing proofs is easy," changed from 1.59 to 2.16. These two questions indicate two important things. First, the scores on the pre-survey showed just how much the students disliked writing proofs after the standard treatment. The scores of 1.38 and 1.59 rate very closely to "strongly disagree". These feelings show that the standard treatment for teaching proofs left the students with a strong dislike for writing proofs and the idea that writing proofs is very difficult as well. This result indicates that perhaps the standard treatment did not do enough to help the students be successful.

The second major result that can be seen from this data is that the proof mapping method helped to move the students' opinions toward the positive direction. Although the post-survey mean scores of 1.91 and 2.16 are still disagreements with the statement, they were much less severe than the first survey. Since this is the case, it seems that the strong dislike for proofs after the standard treatment is very difficult to overcome. Even though the student gained a lot of success using the proof mapping method, they still disliked the process of writing a proof and their dislike for proof writing was softened during the intervention. This also indicates that perhaps a better method for teaching proofs would be to begin with the proof mapping technique initially to try to avoid some of the very discouraged feeling that the students felt using the standard treatment. It would be interesting to see if the survey results on a post-survey of students that had never seen the standard treatment method were higher that the post-survey results of students that did use the standard method. This would be possible to test by using two comparable geometry classes and having one class use only the standard method and then fill out a survey while the other class used only the mapping method and then completed the same survey. It would then be practical to have the first class then switch to the mapping method and the second class learn the standard method and measure the attitudes again. This could show whether or not the initial negative feelings would be consistent regardless of the method chosen to teach proofs or if the mapping method could prevent some of the negative feelings.

The most telling survey statement was number nine. This statement was, “I feel confident in my ability to write proofs.” The students mean scores on this statement changed from 2.18 to 2.75. So this statement almost had students change their opinion entirely. The pre-survey score of 2.18 translated to a “disagree” and the post-survey of 2.78 was much closer to “neither agree nor disagree.” This was a promising result that indicated that the students still thought proofs were difficult but that they were less convinced that they could not write a proof. Even still, these results show that the effects of proof mapping on students' confidence are constructive and worth pursuing further. The students' increased confidence also led them to be slightly more willing to help their classmates work through proofs as well. I saw about a 10% increase in the number of students that were voluntarily willing to help a classmate on a proof during our proof mapping activities compared to activities using the standard treatment. So it appears that students are more willing to help other classmates after they learn the proof map method, which does address one of my sub-questions ("Are students more likely to help their classmates with proofs after learning the proof mapping method ?"), but given the short period of time to allow the students' confidence to develop and ability to increase, more data should be collected to see if this would be a consistent trend using more students and perhaps a longer time period.

All of the reasons and data above seem to provide a clear answer to the sub-question "Can Solow's forward-backward method for writing proofs be successfully adapted to meet the needs of high school students?" The simple answer seems to be "yes." The students had gains in their ability to see relationships between ideas using the mapping method, their ability to write an accurate geometric proof, and their confidence in their ability to write a proof. The proof mapping method seems to be a successful adaptation of the forward-backward method for writing proofs because of the benefits for the students. One of the main reasons for attempting to adapt this method into a visual diagram was to de-emphasize the load on students' short term and working memories that could otherwise have hindered their ability to keep the information organized and accessible in their minds. This adaptation led to an investigation into the effects of the proof mapping method on memory and the effects of memory on this method.

Proof Maps and Memory

One of my sub questions “Does students’ ability to memorize theorems affect their ability to write geometric proofs?” was difficult to test. The original intention was to give one test to the students that asked them to write three proofs without using any list of theorems or postulates and then give the same test later with a list of theorems to see if their scores improved. It was decided that using the same test could affect the results on the second test. A seemingly comparable test was created to try to keep the results as accurate as possible so that even if the students went home after the first test to try to figure out the proofs they were unsure of the second test would be different enough to keep the data accurate. There was an issue with the comparable test, however. Figure 10 below shows a side by side comparison of the first and second test questions. It was found that two questions were just not as comparable as they were intended to be.

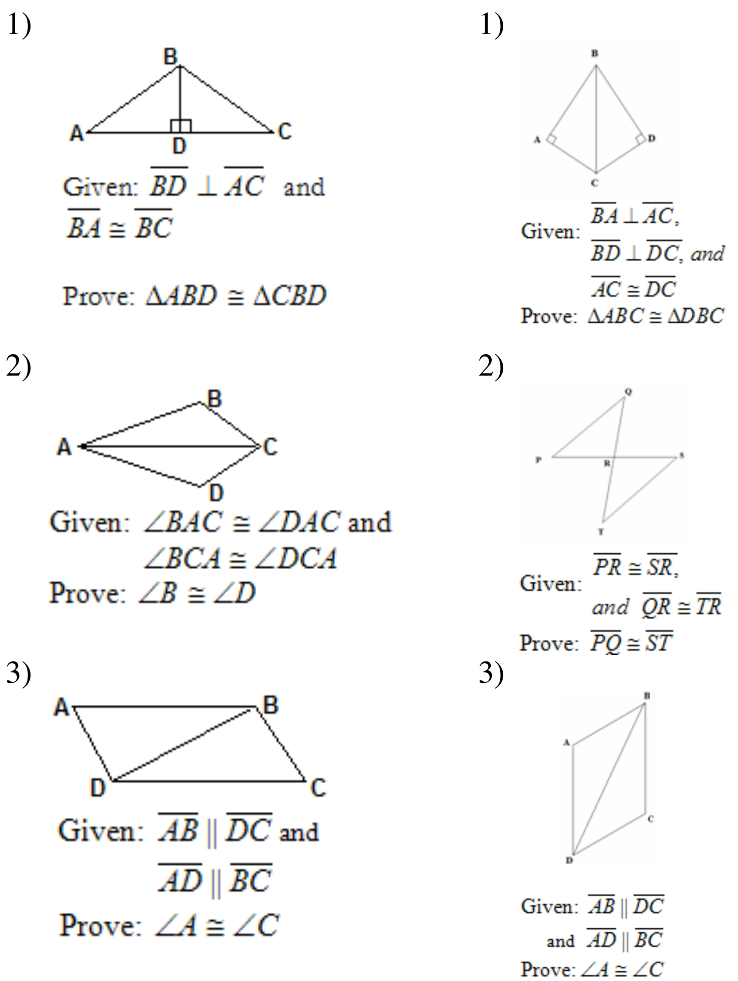


Figure 10. *Note questions one and two require the same number of steps to prove but there are different elements in the first two questions that add difficulty on the second test*.

From the figure, it can be seen that question one on each test uses the Hypotenuse-Leg theorem to prove it but the second test requires the students to use that the segments are perpendicular twice or to indicate two separate facts from this information which added an element of difficulty. Question two on the first test uses the Angle-Side-Angle postulate to prove and on the second test it uses the Side-Angle-Side postulate but the real difficulty came on the second test when the students needed to use the vertical angles theorem instead of the reflexive property which they found to be more difficult as many students justified this step using the reflexive property anyway. Question three was the same question and did show a statistically significant increase in score; however, having only one question shows that the idea that the students' ability to memorize theorems can affect their ability is just insufficient to reach a conclusion. This initial result indicates that memory may have an effect on performance but there is not enough data to know about this conclusively.

Although it was difficult to see if the students' memories affected their ability to write proofs, the proof mapping method did seem to affect the students' ability to memorize at least some of the theorems. One of the other extra benefits of the proof mapping method was the structured rehearsal of practicing all the names of the theorems that could be used to prove that two triangles are congruent. Whenever we asked the key question "How do I prove any two triangles are congruent?" many more students were able to participate in answering the question because they had practiced the answers so much. When we did a proof map as a class and students instructed me on what to write there always seemed to be a race over who could list all six methods first or who could get the last method before anyone else. It was not often that the students got stuck after we had done the first week of proof mapping. They practiced using All Parts Congruent, Side-Side-Side postulate, Side-Angle-Side postulate, Angle-Side-Angle postulate, the Hypotenuse Leg theorem, and the Angle-Angle-Side theorem so much that this first backward step became trivial to many students. They were given the opportunity to practice memorizing the theorems in context and that helped them remember what they were and how to use them. This is a substantial result for many teachers because the ability to know the theorems if the first step in learning how to apply them.

Final Thoughts about the Standard Treatment vs. Proof Mapping

The purpose of this research study was to investigate and address some of the negative or difficult aspects associated with the standard method for teaching and writing geometric proofs. Unlike a high school Algebra class, where students can find success in learning or memorizing an algorithm or copying some template or formula to solve each type of problem, geometry and specifically geometric proofs do not provide such a luxury. The process for writing an accurate proof is not as simple as finding a pattern, shortcut, or algorithm and perhaps that is why so many students dislike the proof writing process- because it challenges their comfort level in mathematics. Geometric proofs require that the students use their logical thinking abilities and defend their statements explicitly— far more work than simply showing one's steps for solving an algebraic equation. Since this is the case, geometry is often the class that causes many students to decide that they are "not good at math," thus preventing them from exploring mathematics further. Geometry also sets the foundation for using logical thinking and reasoning that will later help them understand mathematics at a deeper level— beyond formulas and procedures.

The proof mapping method was found to be a promising technique to address these issues as it provided the students with the ability to write more steps more frequently, increase their knowledge of how geometric properties are connected, write more accurate geometric proofs, and learn geometric theorems in context (beyond pure memorization). Although no single method for teaching geometric proofs will have all the answers, the benefits of proof mapping should be investigated as a tool to help students find a level of success in writing proofs and in geometry that might otherwise discourage the students from pursuing mathematics further. Writing a geometric proof is far more complex than the standard method seems to demonstrate, there is a long thought process and a great deal of consideration that goes into writing a proof and, thus, students should be given a tool that helps them learn to work through this process and think through each possibility as well. While there may never be a clear algorithm for writing geometric proofs that so many students seem to want, the proof mapping technique can provide a sense of structure for a very chaotic and complex process.

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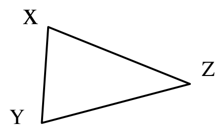
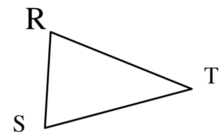
Appendix A: Posttests

Posttest

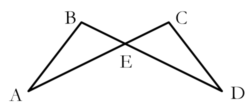
1. **PROVE**:





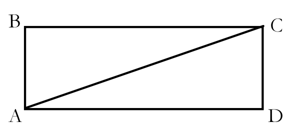
If $\overline{XZ} \cong \overline{RT}$, $\overline{YZ} \cong \overline{ST}$, and $\angle XZY \cong \angle RTS$, then $\Delta XYZ \cong \Delta RST$.

2. **PROVE**:



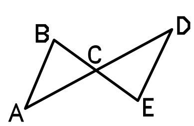
If $\overline{BE} \cong \overline{CE}$ and $\overline{AE} \cong \overline{DE}$, then $\Delta ABE \cong \Delta DCE$.

3. **PROVE**:



If $\overline{BC} \cong \overline{DA}$ and $\overline{BC} \parallel \overline{AD}$, then $\Delta ABC \cong \Delta CDA$.

4. **PROVE**:



If C is the midpoint of $\overline{AD}$ and $\angle A \cong \angle D$, then C is the midpoint of $\overline{BE}$.

Appendix B: Pre and Post-Survey

For each statement below rate your level of agreement on a scale of 1 to 5 with 1 being strongly disagree, 2 being disagree, 3 being neither agree nor disagree, 4 being agree, and 5 being I strongly agree.

	Strong Dis.	Dis.	Not Sure	Agree	Strongly Agree
1) I like writing geometric proofs.	1	2	3	4	5
2) I think writing a proof is easy.	1	2	3	4	5
3) I like writing proofs because they are challenging.	1	2	3	4	5
4) I believe that everyone can be good at proofs if they try hard.	1	2	3	4	5
5) I think geometric proofs help me think about geometry in different ways.	1	2	3	4	5
6) I know I can be successful in math if I want to be.	1	2	3	4	5
7) I would like to learn more about how to write geometric proofs.	1	2	3	4	5
8) I think it would be useful to see different ways of doing the geometric proofs.	1	2	3	4	5
9) I am confident in my own ability to write geometric proofs.	1	2	3	4	5
10) I would feel confident enough to teach another student how to do geometric proofs.	1	2	3	4	5

11) What is your favorite thing about mathematics? Why?

12) What is your least favorite thing about mathematics? Why?

13) When you think about math classes, what is your biggest strength? Why?

14) What is the thing you need to work on most in math classes?

15) Thinking about geometric proofs, what is the most difficult thing about them? Why?

16) For geometric proofs, what do you think would make them easier for you to understand?

Appendix C: Student Work Samples

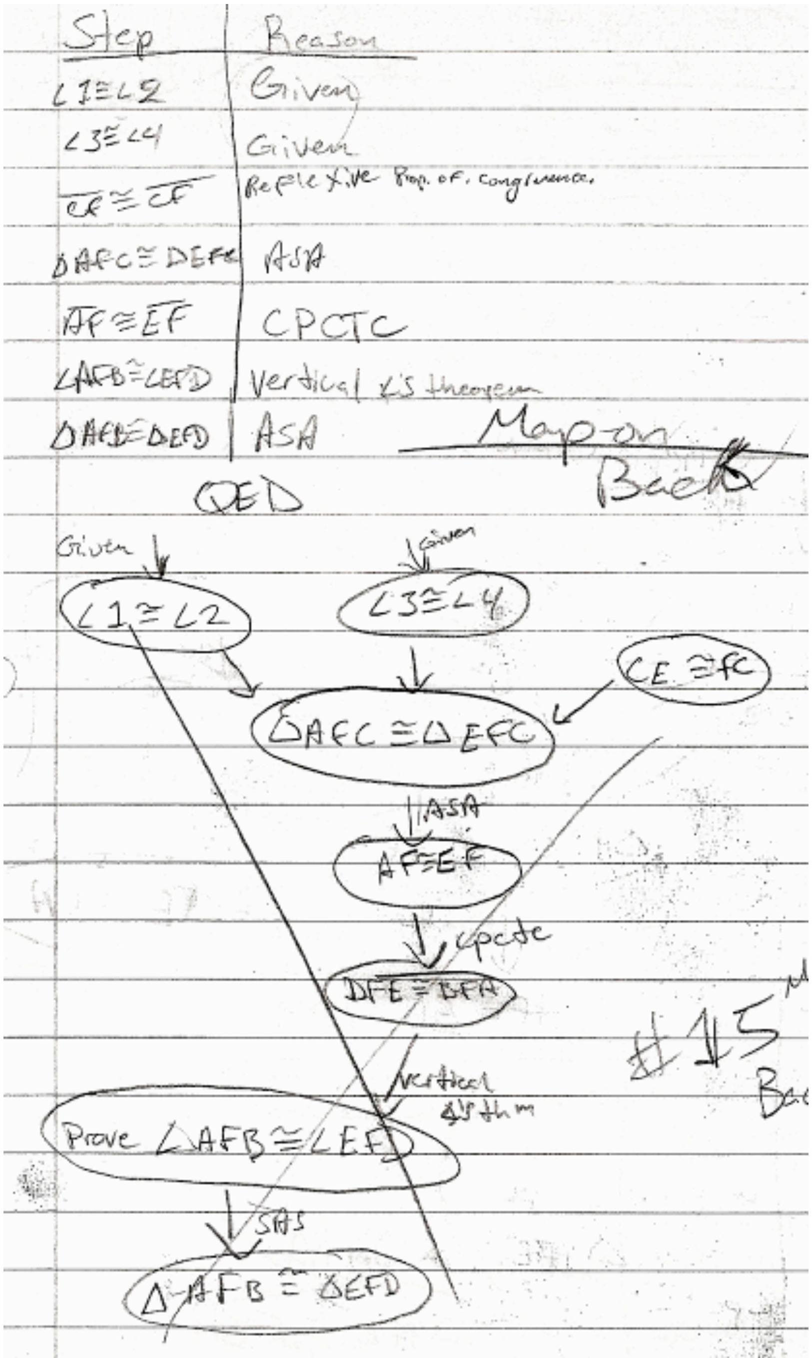


Figure C1. Here we can see the students used the vertical angles theorem as a step following from a statement about congruent segments representing that they needed to say the segments were congruent before they could say that the angles were congruent by the vertical angles theorem. They later corrected the error after a discussion and rewrote an accurate map on the back of the page.

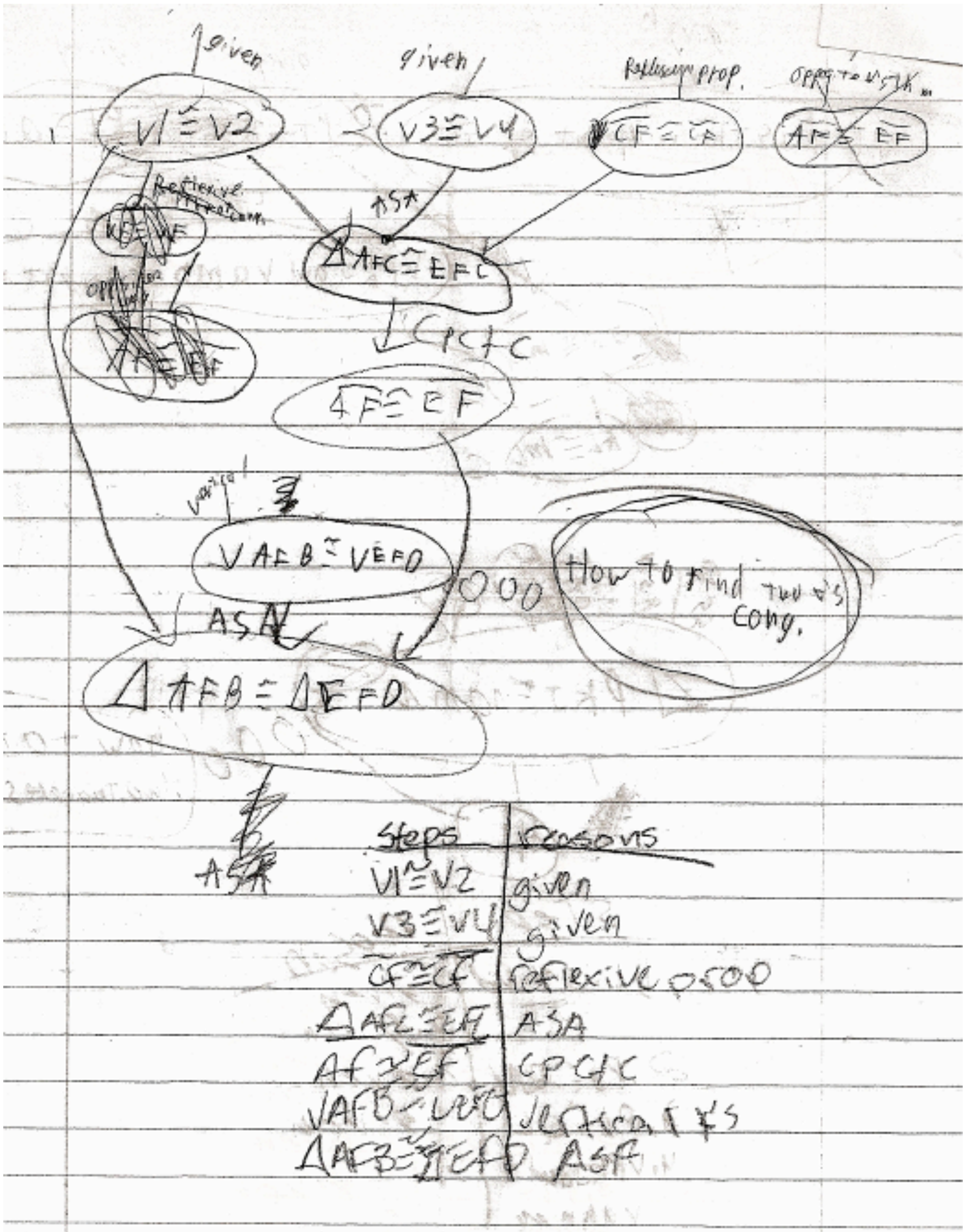


Figure C2. In this work sample the students also incorrectly linked the reflective property to congruent angles even though they did not need to know anything about the angles before they could use the reflexive property. We also discussed this so they crossed it out and fixed the errors.

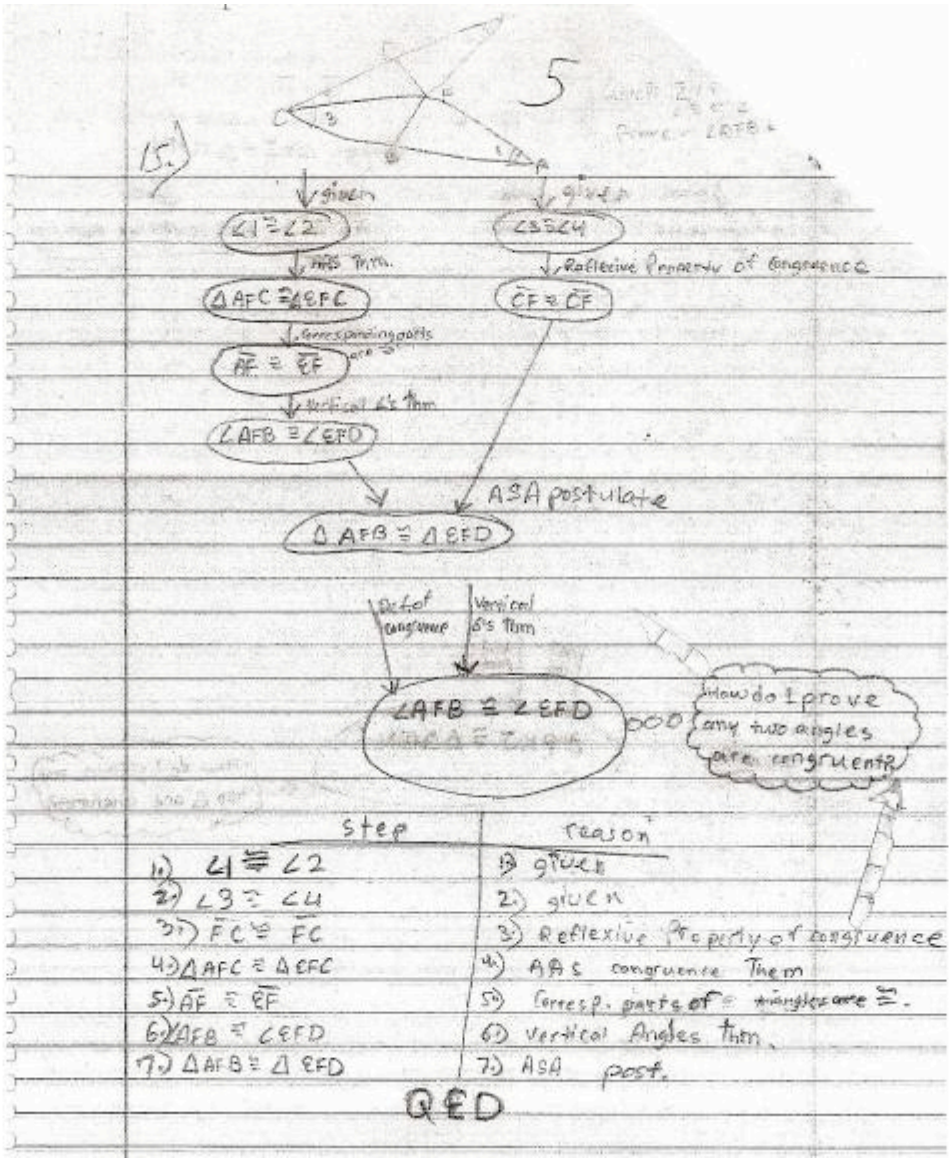


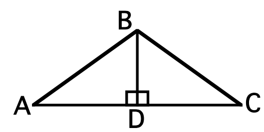
Figure C3. This work sample shows that the students were able to fill in the blanks of the two-column proof but clearly did not understand everything that was required to do this since this proof map again used the congruent angles as a precursor to using the reflexive property and congruent segments as a precursor to using the vertical angles theorem. Even after we discussed this common error as a class this group did not notice their own error.

Appendix D: Proof Mapping Quizzes With and Without Theorems

Quiz: Proof Mapping (Without Theorems)
Directions: Use the Proof Mapping technique to complete the proofs below. You must show all work for credit.

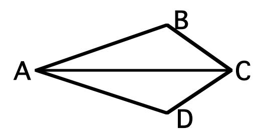
1. **Given**: $\overline{BD} \perp \overline{AC}$ and $\overline{BA} \cong \overline{BC}$

Prove: $\Delta ABD \cong \Delta CBD$



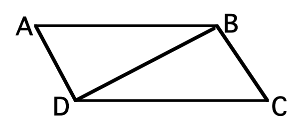
2. **Given**: $\angle BAC \cong \angle DAC$ and $\angle BCA \cong \angle DCA$

Prove: $\angle B \cong \angle D$



3. **Given**: $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$

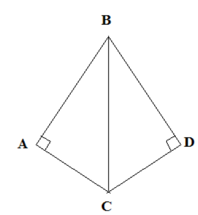
Prove: $\angle A \cong \angle C$



Quiz: Proof Mapping (With Theorems)

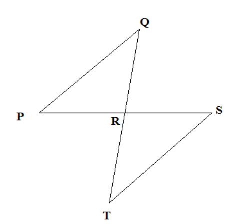
1. **Given**: $\overline{BA} \perp \overline{AC}$, $\overline{BD} \perp \overline{DC}$ and $\overline{AC} \cong \overline{DC}$

Prove: $\Delta ABC \cong \Delta DBC$



2. **Given**: $\overline{PR} \cong \overline{SR}$, *and* $\overline{QR} \cong \overline{TR}$

Prove: $\overline{PQ} \cong \overline{ST}$



3. **Given**: $\overline{AB} \parallel \overline{DC}$ and $\overline{AD} \parallel \overline{BC}$

Prove: $\angle A \cong \angle C$