Odd-Even Game: Player A ____________ Player B _____________

A game is fair if all the players are equally likely to win. Play the following game with a partner twice. Before you play, read the directions and decide if you think the game is fair or unfair.

RULES: **Take turns rolling two number cubes. Find the product of the two numbers. If the product is even, Player A scores a point. If the product is odd, Player B scores a point. After 15 rolls each, the player with more points wins.**

Game 1

#	Product	Player A	Player B
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			
16			
17			
18			
19			
20			
21			
22			
23			
24			
25			
26			
27			
28			
29			
30			
	Total		

Game 2

#	Product	Player A	Player B
1			
2			
3			
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			
14			
15			
16			
17			
18			
19			
20			
21			
22			
23			
24			
25			
26			
27			
28			
29			
30			
	Total		

After playing this game twice do you think the game is fair or unfair?

____

Complete the product chart

	1	**2**	**3**	**4**	**5**	**6**
1	1	2	3			
2	2	4				
3						
4						
5						
6						

What is the total # of even products? ____

What is the total # of odd products? ____

Find *P*(even) ____ *P*(odd) ____

What kind of information does the above table provide?

How can this information help you calculate the probability of an event?

CONCEPT DEVELOPMENT

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Evaluating Statements About Probability

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Beta Version

Evaluating Statements About Probability

MATHEMATICAL GOALS

This lesson unit addresses common misconceptions relating to probability of simple and compound events. The lesson will help you assess how well students understand concepts of:

- Equally likely events.
- Randomness.
- Sample sizes.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics*:

7SP: Investigate chance processes and develop, use, and evaluate probability models

This lesson also relates to the following *Standards for Mathematical Practice* in the *Common Core State Standards for Mathematics*:

2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.

INTRODUCTION

This lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task that is designed to reveal their current understandings and difficulties. You then review their work, and create questions for students to answer in order to improve their solutions.
- After a whole-class introduction, pairs of students work together to justify or refute mathematical statements. They do this using their own examples and counterexamples. Students then explain their reasoning to another group of students.
- In a whole-class discussion students review the main mathematical concepts of the lesson.
- Students return to their original assessment tasks, and try to improve their own responses.

MATERIALS REQUIRED

- Each individual student will need two copies of the assessment task *Are They Correct*?, a mini-whiteboard, a pen, and an eraser.
- Each pair of students will need a copy of the sheet *True, False, or Unsure*?, (cut up into cards), a large sheet of paper for making a poster, and a glue stick.
- There is a projector resource to support a whole-class discussion.

TIME NEEDED

15 minutes before the lesson and a 1-hour lesson. Timings are approximate and will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *Are They Correct?* (15 minutes)

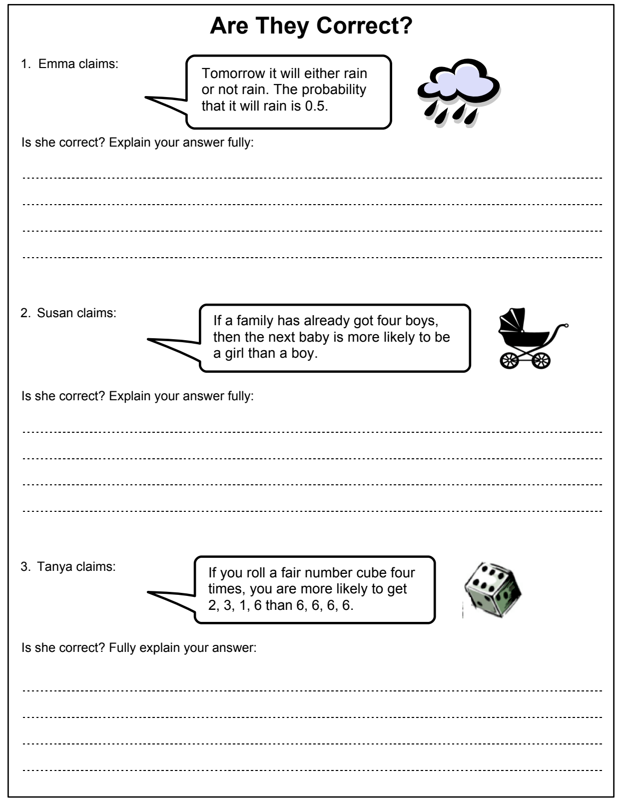
Set this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work. You will then be able to find out the kinds of difficulties students have with it. You will then be able to target your help more effectively in the follow-up lesson.

Give out the assessment task *Are They Correct?*

Briefly introduce the task and help the class to understand each problem.

> *Read through each statement and make sure you understand it.*
>
> *Try to answer each question as carefully as you can.*
>
> *Show all your work so that I can understand your reasoning.*

It is important that, as far as possible, students are allowed to answer the questions without your assistance.



Are They Correct?

1. Emma claims: Tomorrow it will either rain or not rain. The probability that it will rain is 0.5.

Is she correct? Explain your answer fully:

2. Susan claims: If a family has already got four boys, then the next baby is more likely to be a girl than a boy.

Is she correct? Explain your answer fully:

3. Tanya claims: If you roll a fair number cube four times, you are more likely to get 2, 3, 1, 6 than 6, 6, 6, 6.

Is she correct? Fully explain your answer:

Students should not worry too much if they cannot understand or do everything, because in the next lesson they will engage in a similar task, which should help them. Explain to students that by the end of the next lesson, they should expect to answer questions such as these confidently. This is their goal.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem solving approaches.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores, and will distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given on the next page. These have been drawn from common difficulties observed in trials of this unit.

We suggest that you write a list of your own questions, based on your students' work, using the ideas that follow. You may choose to write questions on each student's work. If you do not have time to do this, select a few questions that will be of help to the majority of students. These can be written on the board at the end of the lesson.

Common issues:	**Suggested questions and prompts:**
Q1. Student assumes that both outcomes are equally likely	• What factors affect whether it will rain tomorrow? • What does a probability of 0.5 mean?
Q2. Student assumes that later random events 'compensate' for earlier ones For example: The student argues that if there are already four boys in the family, the next will is likely to be a girl.	• What is the probability that the baby will be a girl? • Does the fact that there are already four boys in the family affect the sex of the next child?
Q3. Student relies on their own experience For example: The student states they have never thrown four sixes in a row.	• Is it more difficult to throw a six than a two? • Is it more difficult to throw a six, then another six or a two, then a three?

SUGGESTED LESSON OUTLINE

Whole-class interactive introduction (10 minutes)

Give each student a mini-whiteboard, a pen, and an eraser.

Display slide P-1 of the projector resources.

Two bags of jellybeans
I have two bags. Both contain red and yellow jellybeans. There are more red jellybeans in bag **A** than in bag **B**. If I choose one jellybean from each bag I am more likely to choose a red one from bag **A** than from bag **B**.

Ensure the students understand the problem:

Your task is to decide if the statement is true.

Once you have made a decision you need to convince me.

Allow students a few minutes to think about the problem individually, then a further few minutes to discuss their initial ideas in pairs. Ask students to write their explanations on their whiteboard.

If students are unsure, encourage them to think of a simple experiment that could simulate the statement:

Do you know how many red and yellow jellybeans are in each bag? Give me an example of the numbers of jellybeans in each bag. Draw a picture of the situation.

Can you think of a situation for which the statement is true? [For example, two red jellybeans and one yellow jellybean in bag A and one red jellybean and one yellow jellybean in bag B.]

Can you think of a situation for which the statement is false? [For example, two red jellybeans and three yellow jellybeans in bag A and one red jellybean and one yellow jellybean in bag B.]

Ask students to show you their mini-whiteboards. Select two or three students with different answers to explain their reasoning on the board. Encourage the rest of the class to comment.

Then ask:

Chen, can you rewrite the statement so that it is always true?

Carlos, do you agree with Chen's explanation? Put Chen's explanation into your own words.

Does anyone have a different statement that is also always true?

This statement highlights the misconception that students often think the results of random selection are dependent on numbers rather than ratios.

Collaborative activity (20 minutes)

Organize the class into pairs of students.

Give each pair a copy of *True, False, or Unsure*?, a large piece of paper for making a poster, and a glue stick.

Ask students to divide their paper into two columns: one for statements they think are true, and the other for statements they think are false.

Ask students to take each statement in turn:

Select a card and decide whether it is a true or false statement.

Convince your partner of your decision.

It is important that you both understand the reasons for the decision. If you don't agree with your partner, explain why. You are both responsible for each other's learning.

If you are both happy with the decision, glue the card onto the paper. Next to the card, write reasons to support your decision.

Put to one side any cards you are unsure about.

You may want to use slide P-2 of the projector resource to display these instructions

You have two tasks during small group work: to make a note of student approaches to the task, and to support student problem solving.

Make a note of student approaches to the task

Notice how students make a start on the task, whether they get stuck, and how they respond if they do come to a halt. For example, are students drawing diagrams, working out probabilities, or simply writing a description? As they work on the task, listen to their reasoning carefully and note misconceptions that arise for later discussion with the whole class.

Support student problem solving

Try not to make suggestions that move students towards a particular approach to this task. Instead, ask questions to help students clarify their thinking. If several students in the class are struggling with the same issue, write a relevant question on the board and hold a brief whole-class discussion.

Here are some questions you may want to ask your students:

Card A: This statement addresses the misconception that probabilities give the proportion of outcomes that *will* occur.

Is it possible to get five sixes in a row with a fair six-sided number cube?

Is it more difficult to roll a six than, say, a two?

Card B: This statement addresses the misconception that 'special' events are less likely than 'more representative' events. Students often assume that selecting an 'unusual' letter, such as W, X, Y or Z is a less likely outcome.

Is the letter X more difficult to select than the letter T?

Are the letters W and X more difficult to select than the letters D and T?

Card C: This statement addresses the misconception that later random events 'compensate' for earlier ones.

Does the coin have a memory?

Card D: This statement addresses the misconception that all outcomes are equally likely, without considering that some are much more likely than others.

Is the probability of a local school soccer team beating the World Cup champions $\frac{1}{3}$*?*

Card E This statement addresses the misconception that all outcomes are equally likely, without considering that some are much more likely than others. Students often simply count the different outcomes.

Are all three outcomes equally likely? How do you know? How can you check your answer?

What are all the possible outcomes when two coins are tossed? How does this help?

Card F: This statement addresses the misconception that the two outcomes are equally likely.

How can you check your answer?

In how many ways can you score a three? In how many ways can you score a two?

Card G: This statement addresses the misconception that probabilities give the proportion of outcomes that *will* occur.

When something is certain, what is its probability?

What experiment could you do to check if this answer is correct? [One student writes the ten answers e.g. false, true, true, false, true, false, false, false, true, false. Without seeing these answers the other student guesses the answers].

Card H: This statement addresses the misconception that the sample size is irrelevant. Students often assume that because the probability of one head in two coin tosses is $\frac{1}{2}$, the probability of n heads in $2n$ coin tosses is also $\frac{1}{2}$.

Is the probability of getting one head in two coin tosses $\frac{1}{2}$*? How do you know?*

Show me a possible outcome if there are four coin tosses. Show me another. How many possible outcomes are there? How many outcomes are there with two heads?

Sharing work (5 minutes)

As students finish the task, ask them to compare work with a neighboring pair.

Check which answers are different.

A member of each group needs to explain their reasoning for these answers. If anything is unclear, ask for clarification.

Then together consider if you should change any of your answers.

It is important that everyone in both groups understands the math. You are responsible for each other's learning.

Whole-class discussion (10 minutes)

Organize a discussion about what has been learned. Focus on getting students to understand the reasoning, not just checking that everyone produced the same answers.

Ask students to choose one card they are certain is true and to explain why they are certain to the rest of the class. Repeat this with the statements that students believe are false. Finally, as a whole class, tackle the statements that students are not so sure about.

Ben, why did you decide this statement was true/false?

Does anyone agree/disagree with Ben?

Does anyone have a different explanation to Ben's?

In addition to asking for a variety of methods, pursue the theme of listening and comprehending each others' methods by asking students to rephrase each other's reasoning.

Danielle, can you put that into your own words?

You may also want to ask students:

Select two cards that use similar math. Why are they similar? Is there anything different about them? [Students are likely to select cards D and E.]

In trials, students have found card H challenging.

What are the possible outcomes? Have you listed all of the outcomes? Have you listed all the outcomes where there are two heads? What does this show?

Improving individual solutions to the assessment task (10 minutes)

Return the original assessment task *Are they Correct?* to students, together with a blank copy of the task.

Look at your original responses and think about what you have learned this lesson.

Using what you have learned, try to improve your work.

If you have not added questions to individual pieces of work, then write your list of questions on the board. Students should select from this list only the questions they think are appropriate to their own work.

If you find you are running out of time, then you could set this task in the next lesson, or for homework.

SOLUTIONS

Assessment Task: *Are They Correct?*

1. This statement is incorrect. It highlights the misconception that all events are equally likely. There are many factors (e.g., the season) that will influence the chances of it raining tomorrow.
2. *Assuming that the sex of a baby is a random, independent event equivalent to tossing a coin*, the statement is incorrect. It highlights the misconception that later random events can 'compensate' for earlier ones. The assumption is important: there are many beliefs and anecdotes about what determines the sex of a baby, but 'tossing a coin' turns out to be a reasonably good model[1].
3. This statement is incorrect. This highlights the misconception that 'special' events are less likely than 'more representative' events.

Collaborative Activity: *True, False, or Unsure*

A. If you roll a six-sided number cube, and it lands on a six more than any other number, then the number cube must be biased.

False. This statement addresses the misconception that probabilities give the proportion of outcomes that **will** occur. With more information (**How many** times was the cube rolled? **How many** more sixes were thrown?) more advanced mathematics could be used to calculate the **probability** that the dice was biased, but you could never be 100% certain.

B. When randomly selecting four letters from the alphabet, you are more likely to come up with

D, T, M, J than W, X, Y, Z.

False. This highlights the misconception that 'special' events are less likely than 'more representative' events. Students often assume that selecting the 'unusual' letters W, X, Y and X is less likely.

C. If you toss a fair coin five times and get five heads in a row, the next time you toss the coin it is more likely to show a tail than a head.

False. This highlights the misconception that later random events 'compensate' for earlier ones. The statement implies that the coin has some sort of 'memory'. People often use the phrase 'the law of averages' in this way.

D. There are three outcomes in a soccer match: win, lose, or draw. The probability of winning is therefore $\frac{1}{3}$.

False. This highlights the misconception that all outcomes are equally likely, without considering that some are much more likely than others. The probabilities are dependent on the rules of the game and which teams are playing.

---

[1] See, for example, http://www.bbc.co.uk/news/magazine-12140065

E. When two coins are tossed there are three possible outcomes: two heads, one head, or no heads. The probability of two heads is therefore $\frac{1}{3}$.

False. This highlights the misconception that all outcomes are equally likely, without considering that some are much more likely than others. There are four equally likely outcomes: HH, HT, TH, TT. The probability of two heads is $\frac{1}{4}$.

F. Scoring a total of three with two number cubes is twice as likely as scoring a total of two.

True. This highlights the misconception that the two outcomes are equally likely. To score three there are two outcomes, 1,2 and 2,1, but to score two there is only one outcome, 1,1.

G. In a 'true or false?' quiz with ten questions, you are certain to get five correct if you just guess.

False. This highlights the misconception that probabilities give the exact proportion of outcomes that will occur. If a lot of people took the quiz, you would expect the mean score to be *about* 5, but the individual scores would vary.

Probabilities do not say for certain what will happen, they only give an indication of the likelihood of something happening. The only time we can be certain of something is when the probability is 0 or 1.

H. The probability of getting exactly two heads in four coin tosses is $\frac{1}{2}$.

False. This highlights the misconception that the sample size is irrelevant. Students often assume that because the probability of one head in two coin tosses is $\frac{1}{2}$, then the probability of n heads in $2n$ coin tosses is also $\frac{1}{2}$. In fact the probability of two out of four coin tosses being heads is $\frac{6}{16}$.

This can be worked out by writing out all the sixteen possible outcomes:

HHHH, HHHT, HHTH, HTHH, THHH, TTTT, TTTH, TTHT, THTT, HTTT, HHTT, HTTH, TTHH, THTH, HTHT, THHT.

This may be calculated from Pascal's triangle:

	1		
2 coins	1 **2** 1	4 outcomes	Probability(1 head) =
	1 3 3 1		
4 coins	1 4 **6** 4 1	16 outcomes	Probability(2 heads)
	1 5 10 10 5 1		
6 coins	1 6 15 **20** 15 6 1	64 outcomes	Probability(3 heads)

Students are not expected to make this connection!

Are They Correct?

1. Emma claims:

> Tomorrow it will either rain or not rain. The probability that it will rain is 0.5.

Is she correct? Explain your answer fully:

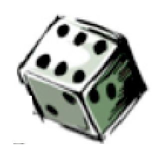
2. Susan claims:

> If a family has already got four boys, then the next baby is more likely to be a girl than a boy.

Is she correct? Explain your answer fully:

3. Tanya claims:

> If you roll a fair number cube four times, you are more likely to get 2, 3, 1, 6 than 6, 6, 6, 6.



Is she correct? Fully explain your answer:

Card Set: True, False Or Unsure?

A. If you roll a six-sided number cube, and it lands on a six more than any other number, then the number cube must be biased. 	B. When randomly selecting four letters from the alphabet, you are more likely to come up with D, T, M, J than W, X, Y, Z.
C. If you toss a fair coin five times and get five heads in a row, the next time you toss the coin it is more likely to show a tail than a head.	D. There are three outcomes in a soccer match: win, lose, or draw. The probability of winning is therefore $\frac{1}{3}$. 
E. When two coins are tossed there are three possible outcomes: two heads, one head, or no heads. The probability of two heads is therefore $\frac{1}{3}$.	F. Scoring a total of three with two number cubes is twice as likely as scoring a total of two. 
G. In a “true or false?” quiz with ten questions, you are certain to get five correct if you just guess. 	H. The probability of getting exactly two heads in four coin tosses is $\frac{1}{2}$. 

Two bags of jellybeans

I have two bags. Both contain red and yellow jellybeans.

There are more red jellybeans in bag **A** than in bag **B**.

If I choose one jellybean from each bag I am more likely to choose a red one from bag **A** than from bag **B**.

True, False or Unsure?

- Take turns to select a card and decide whether it is a true or false statement.

- Convince your partner of your decision.

- It is important that you both understand the reasons for the decision. If you don’t agree with your partner, explain why. You are both responsible for each other’s learning.

- If you are both happy with the decision, glue the card onto the paper. Next to the card, write reasons to support your decision.

- Put to one side any cards you are unsure about.

Mathematics Assessment Project

CLASSROOM CHALLENGES

This lesson was designed and developed by the
Shell Center Team
at the
University of Nottingham
Malcolm Swan, Nichola Clarke, Clare Dawson, Sheila Evans
with
Hugh Burkhardt, Rita Crust, Andy Noyes, and Daniel Pead

It was refined on the basis of reports from teams of observers led by
David Foster, Mary Bouck, and Diane Schaefer
based on their observation of trials in US classrooms
along with comments from teachers and other users.

This project was conceived and directed for
MARS: Mathematics Assessment Resource Service
by
Alan Schoenfeld, Hugh Burkhardt, Daniel Pead, and Malcolm Swan
and based at the University of California, Berkeley

We are grateful to the many teachers, in the UK and the US, who trialed earlier versions
of these materials in their classrooms, to their students, and to
Judith Mills, Carol Hill, and Alvaro Villanueva who contributed to the design.

This development would not have been possible without the support of
Bill & Melinda Gates Foundation
We are particularly grateful to
Carina Wong, Melissa Chabran, and Jamie McKee

Modifying the Sample Space Name:

Most states have some kind of lottery. How do the chances of winning a lottery when a winner has to match three numbers compare to when the winner has to match seven numbers? How do the chances of catching a particular kind of fish in a pond change just after it is filled with more fish? In this lesson, you will think about the size of the **sample space**, that is, the collection of all possible outcomes of an event. Think about these questions as you work today:

How has the "whole" or total changed?

How has the "portion" or part we are interested in changed?

Has the event become more or less likely?

2-30. Your team will be given a bag containing a set of colored blocks or counters. Each team will receive a bag that is identical to yours.

a. Look at the blocks in your bag. If you were to reach into the bag and select one block without looking, what is the likelihood that it would be:

i. Red? ____ *ii.* Green? ___

iii. Blue? ____ *iv.* Orange? ___

b. Do your answers for part (a) represent theoretical or experimental probabilities? Justify your response.

2-31. If you were to select one block from the bag 12 times, replacing the block you drew between each selection, how many of those times would you expect to have selected a blue block?

What if you drew 24 times?

Discuss both situations with your team and explain your answers.

2-32. DOUBLING BAGS

Now imagine that you and another team have combined the blocks from both of your bags into one bag.

a. Do you think the larger sample space will change the likelihood of drawing different colored blocks?

Discuss this with your team and be ready to explain your ideas to the class.
b. Get a second bag of blocks from your teacher and merge the contents of both bags. How many total blocks are there in the bag now?

How many are there of each color?

c. Work with your team to find the theoretical probability for selecting each color of block in the combined bags.

d. Has the probability for drawing each different colored block changed?

How do your answers for part (c) above compare to the theoretical probabilities that you calculated for the original bag in problem 2-30?

With your team, make sense of how the probability for drawing a blue block compares before and after combining the bags.

e. If you were to make 12 draws from the combined bag, again replacing the block between draws, how many times would you expect to draw a blue block?

Explain why your answer makes sense.

2-33. In problems 2-30 through 2-32, even though you combined bags and changed the number of selections you made, the probability of drawing a blue block remained the same.

a. Do you think the probabilities would change if you combined three bags?
Why or why not?

b. What change do you think you could make in order to increase the chances of choosing a blue block? Explain your reasoning.

PROBLEM SOLVING

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Modeling Conditional Probabilities 1: Lucky Dip

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley

Beta Version

Modeling Conditional Probabilities 1: *Lucky Dip*

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to:

- Understand conditional probability.
- Represent events as a subset of a sample space using tables and tree diagrams.
- Communicate their reasoning clearly.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Mathematical Practices* in the *Common Core State Standards for Mathematics*:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content* in the CCSS:

S-CP: Understand independence and conditional probability and use them to interpret data.
S-MD: Calculate expected values and use them to solve problems.

INTRODUCTION

This lesson unit is structured in the following way:

- Before the lesson, students work individually on an assessment task that is designed to reveal their current understanding and difficulties. You then review their work and create questions for students to answer in order to improve their solutions.
- At the start of the lesson, students work alone answering your questions about the same problem. Students are then grouped and engage in a collaborative discussion of the same task.
- In the same small groups, students are given sample solutions to analyze and evaluate.
- Finally, in a whole-class discussion, students explain and compare the alternative solution strategies they have seen and used.
- Finally, students review what they have learnt.

MATERIALS REQUIRED

- Each student will need a copy of the assessment task, *Lucky Dip*, a copy of the review sheet, *How did you work?*, a mini-whiteboard, a pen, and an eraser.
- Each small group of students will need a large sheet of paper for making a poster, a felt-tipped pen, and enlarged copies of the *Student Sample Responses*.
- There are some projector resources to support whole class discussion.
- You will need a bag, and some black and white balls (or some substitute) for a class demonstration.

TIME NEEDED

20 minutes before the lesson, a 1- hour lesson, and 10 minutes in a subsequent lesson. Exact timings will depend on the needs of the class.

BEFORE THE LESSON

Assessment task: *Lucky Dip* task (20 minutes)

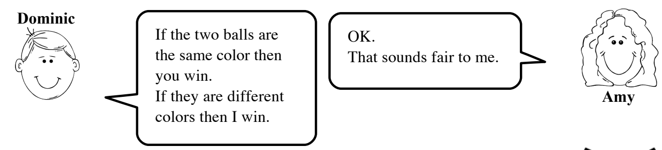
Set this task, in class or for homework, a few days before the formative assessment lesson. This will give you an opportunity to assess the work, and to find out the kinds of difficulties students have with it You will then be able to target your help more effectively in the follow-up lesson.

Give each student a copy of the assessment task *Lucky Dip*.

Lucky Dip

Dominic has devised a simple game.
Inside a bag he places 3 black and 3 white balls. He then shakes the bag.

He asks Amy to take two balls from the bag without looking.



Is Amy right? Is the game fair?

If Amy is wrong, then who is most likely to win?

Show all your reasoning clearly.

Make sure the class understands the rules of the game by demonstrating it using a bag, and some black and white balls.

Read through the questions and try to answer them as carefully as you can.

It is important that students, as far as possible, are allowed to answer the questions without your assistance.

Students should not worry too much if they cannot understand or do everything, because in the next lesson they will engage in a similar task, which should help them. Explain to students that by the end of the next lesson, they should expect to answer questions such as these confidently. This is their goal.

Students who sit together often produce similar answers, and then when they come to compare their work, they have little to discuss. For this reason, we suggest that when students do the task individually, you ask them to move to different seats. Then at the beginning of the formative assessment lesson, allow them to return to their usual seats. Experience has shown that this produces more profitable discussions.

Assessing students' responses

Collect students' responses to the task. Make some notes on what their work reveals about their current levels of understanding and their different problem solving approaches.

We suggest that you do not score students' work. The research shows that this will be counterproductive, as it will encourage students to compare their scores, and will distract their attention from what they can do to improve their mathematics.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given on the next page. These have been drawn from common difficulties observed in trials of this unit.

We recommend that you write a selection of questions on each student's work. If you do not have time, select a few questions that will be of help to the majority of students. These can be written on the board at the beginning of the lesson.

Common issues	Suggested questions and prompts
Student produces no work.	• Play the game twenty times, using pieces of marked paper instead of balls. Do you think the game is fair? Explain your answer.
Student assumes the game is fair. For example: The student assumes that there are only two outcomes (the balls are the same color or the balls are different colors), so the probabilities are equal.	• Suppose you labeled each ball with a different letter. What are the different combinations you can take out of the bag?
Student does not choose a suitable representation. For example: The student does not use a tree diagram, or a sample space diagram.	• Can you think of a suitable diagram that will show all the possible outcomes? • Can you use a sample space diagram?
Student is confused as to the nature of the "event". For example: The student is concerned over whether it makes a difference to consider taking both balls out at once or taking them out one at a time.	• Does it make a difference whether Amy picks the balls one at a time, rather than at the same time? Explain your answer. • How can you show the different possible outcomes using a diagram?
Student does not recognize dependent probabilities. The student appears to assume that each ball is returned to the bag after it is selected. For example: The probability that the second ball is black is assumed to be independent of the choice of the first ball. So P(both balls black) is assumed to be 0.5×0.5.	• Imagine picking a black ball from the bag. What is the probability of picking a black ball? What has changed? Now you are holding the black ball, what is the probability of picking another black ball?
Student selects the same ball twice in their table of possible outcomes. For example: The student assumes that there are $3 \times 3 = 9$ ways of obtaining two black balls.	• Is it possible to select the same ball twice?
Student presents incomplete or unclear work. For example: The student does not fully label the tree diagram or the sample space diagram.	• Would someone unfamiliar with this type of work understand all your work?
Student correctly answers all the questions. Student needs an extension task.	• How many black balls and how many white balls could you put in the bag to make the game fair? Explain your answer.

SUGGESTED LESSON OUTLINE

Individual work (10 minutes)

Return the assessment task to the students. Give each student a mini-whiteboard, a pen, and an eraser.

Begin the lesson by briefly reintroducing the problem.

If you did not add questions to individual pieces of work, write your list of questions on the board. Students are to select questions appropriate to their own work, and spend a few minutes answering them. If students struggle to identify which questions they should be considering, it may be helpful to give them a printed version of the list of questions and highlight the ones that you want them to focus on.

Recall what we were looking at in a previous lesson. What was the task about?

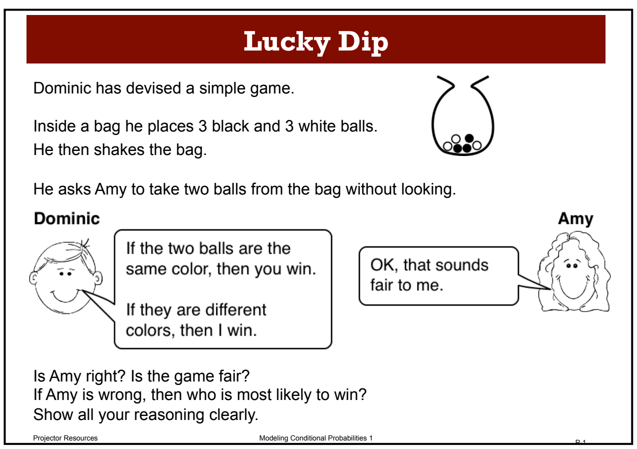
Today we are going to work together to try to improve your initial attempts at this task.

I have had a look at your work, and I have some questions I would like you to think about.

On your own, carefully read through the questions I have written. Use the questions to help you to think about ways of improving your own work.

Use your mini-whiteboards to make a note of anything you think will help to improve your work.

Slide P-1 of the projector resource outlines the rules of the game. To remind students of the rules, you could demonstrate the game using a bag and some balls.



Collaborative small-group work (10 minutes)

Organize the class into small groups of two or three students. Give each group a large, piece of paper, and a felt-tipped pen.

Deciding on a Strategy

Ask students to share their ideas about the task, and plan a joint solution.

I want you to share your work with your group.

Take turns to explain how you did the task and how you now think it could be improved.

Listen carefully to any explanation. Ask questions if you don't understand or agree with the method. (You may want to use some of the questions I have written on the board.)

I want you to plan a joint approach that is better than your separate solutions.

Once students have evaluated the relative merits of each approach ask them to write their strategy on the second side of the poster.

Slide P-2 of the projector resource, *Planning a Joint Solution*, summarizes these instructions.

Planning a Joint Solution

1. Take turns to explain how you did the task and how you now think it could be improved.

2. Listen carefully to explanations.
   – Ask questions if you don't understand.

   – Discuss with your partners:
     • What you like/dislike about your partners' math.
     • Any assumptions your partner has made.
     • How their work could be improved.

3. Once everyone in the group has explained their solution, plan a joint approach that is better than each of the separate solutions.
   – On the second side of your poster or paper write a couple of sentences outlining your plan.

Implementing the Strategy

Students are now to write their joint solution on the front side of the poster.

While students work in small groups you have two tasks: to note different student approaches to the task, and to support student problem solving

You can then use this information to focus a whole class discussion towards the end of the lesson. For example, do students identify all the different possible events clearly? Are students using diagrams to support their answers? In particular, note any common mistakes.

Try not to make suggestions that move students towards a particular approach to the task. Instead, ask questions that help students to clarify their thinking. In particular focus on the strategies rather than the solution. Encourage students to justify their statements.

Look for any groups of students who agree amongst themselves on an incorrect answer or justification. You could ask these students to work with another group, to compare solutions and prompt revision.

You may want to use the questions in the *Common issues* table to support your own questioning. If the whole class is struggling on the same issue, you could write one or two relevant questions on the board and hold a brief whole-class discussion. You could also give any struggling students one of the sample responses.

Whole-class discussion (10 minutes)

You may want to hold a brief whole-class discussion. Have students solved the problem using a variety of methods? Or have you noticed some interesting ways of working or some incorrect methods, if so, you may want to focus the discussion on these. Equally, if you have noticed different groups using similar strategies but making different assumptions you may want to compare solutions.

Collaborative analysis of *Sample Student Responses* (15 minutes)

After students have had enough time to attempt the problem, give each group copies of the three *Sample student responses*, and ask for written comments. This task gives students the opportunity to evaluate a variety of approaches to the task.

You are now going to look at three solutions to the task.

Imagine you are the teacher. Write down your comments on each piece of work.

Try to explain what the student has done.

What mistakes have been made?

What isn't clear about the work?

Slide P-3 of the projector resource, *Evaluating Sample Student Work*, describes how students should work together:

Evaluating Student Sample Responses

1. Imagine you are the teacher and have to assess the student work.
2. Take turns to work through a students' solution.
   - Write your answers on your mini-whiteboards.
3. Explain your answer to the rest of the group.
4. Listen carefully to explanations.
   - Ask questions if you don't understand.
5. Once everyone is satisfied with the explanations, write the answers below the students' solution.
   - Make sure the student who writes the answers is not the student who explained them.

During the paired work, support the students as in the first collaborative activity.

Note similarities and differences between the approaches seen in the *Sample responses* and those students took in the small-group work. Also, check to see which methods students have difficulties in understanding. This information can help you focus the next activity, a whole-class discussion.

Whole-class discussion: comparing different approaches (15 minutes)

Hold a whole-class discussion to consider the different approaches used in the sample work. Focus the discussion on parts of the task students found difficult. Ask the students to compare the different solution methods.

Which approach did you like best? Why?

Which approach did you find most difficult to understand? Why?

To support the discussion, you may want to use the projector resource to display the sample work.

Anna's work appears intuitively correct.

She assumes that there are only two outcomes (that the two balls are the same color or that they are different colors), so that the probabilities are equal.

Anna does not take into account the changes in probabilities once a ball is removed from the bag and not replaced.

Amy could select
Black + black
Black + white
White + black
White + white

There are 2 when the balls are the same color + 2 when the balls are different

THE GAME IS FAIR

Ella draws a sample space in the form of an organized table.

Ella clearly presents her work, however she makes the mistake of including the diagonals. This means the same ball is selected twice. This is not possible, as the balls are not replaced.

2nd ball

1st ball	B1	B2	B3	W1	W2	W3
B1	Amy	A	A	D	D	D
B2	A	A	A	D	D	D
B3	A	A	A	D	D	D
W1	D	D	D	A	A	A
W2	D	D	D	A	A	A
W3	D	D	D	A	A	A

There are 36 equally likely outcomes.

Amy wins 18 times

Dominic wins 18 times

So the game is fair.

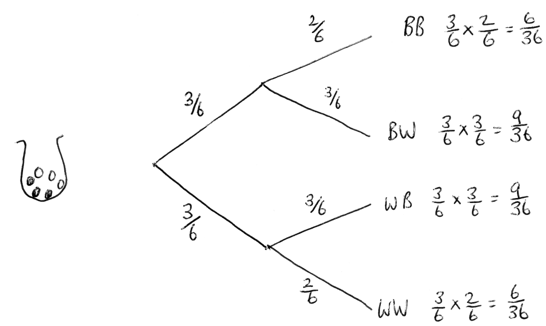
Jordan uses a tree diagram to show the possible outcomes when taking two balls from the bag.

Jordan's work is difficult to follow. He does not label the branches of the tree.

Jordan does not take into account that the first ball is not replaced. When selecting the second ball there are only 5 balls in the bag, so these probability fractions should all have a denominator of 5.

Where does the denominator of 6 come from?

What is the sum of all final probabilities?
What does this tell you about Jordan's work?
[He has made a mistake.]



2 same color $= \frac{6}{36} + \frac{6}{36} = \frac{12}{36} = \frac{1}{3}$

2 different color $= \frac{9}{36} + \frac{9}{36} = \frac{18}{36} = \frac{1}{2}$

Review solutions to *Lucky Dip* (10 minutes)

Give out the sheet *How Did You Work?* and ask students to complete this questionnaire.

Some teachers set this task as homework.

The questionnaire should help students review their progress.

If you have time you may also want to ask your students to read through their original solution and using what they have learned, attempt the task again. In this case, give each student a fresh blank copy of the assessment *Lucky Dip*.

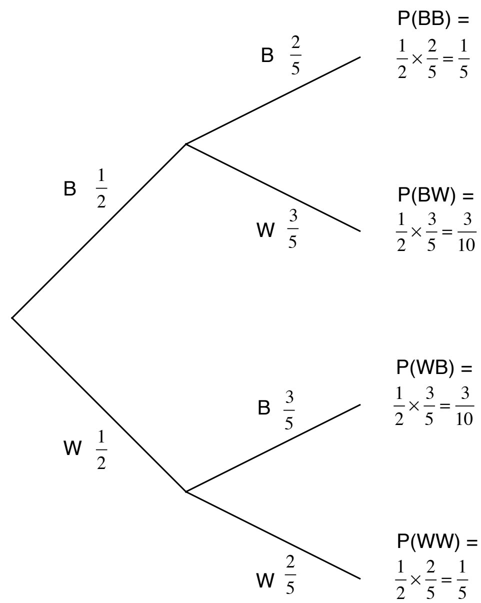
SOLUTIONS

Lucky Dip

Amy is wrong: the game is not fair. In the sample space diagram below, the black balls are labeled B_1, B_2, B_3 and the white balls are labeled W_1, W_2, W_3. Each cell shows one possible, equally likely outcome. The diagonal doesn't show possible outcomes because the same ball cannot be taken out twice.

Amy wins wherever there is a ✓. Dominic wins wherever there is a x. This shows that the probability of Amy winning is $\frac{12}{30}=\frac{2}{5}$ and the probability of Dominic winning is $\frac{18}{30}=\frac{3}{5}$. An alternative representation is the tree diagram.

		First selection					
		B_1	B_2	B_3	W_1	W_2	W_3
Second selection	B_1	-	✓	✓	x	x	x
	B_2	✓	-	✓	x	x	x
	B_3	✓	✓	-	x	x	x
	W_1	x	x	x	-	✓	✓
	W_2	x	x	x	✓	-	✓
	W_3	x	x	x	✓	✓	-

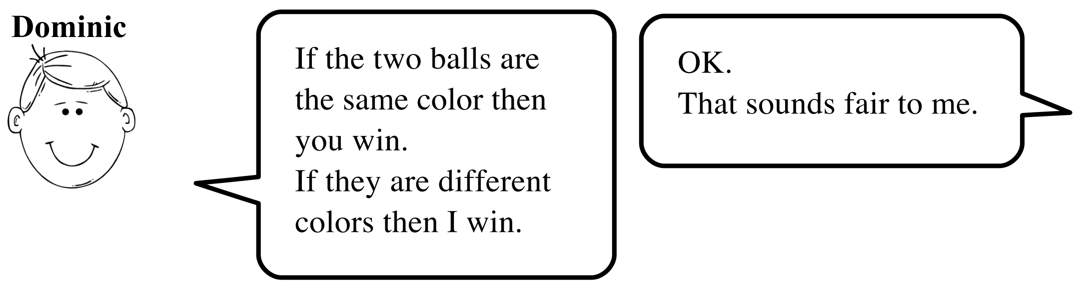


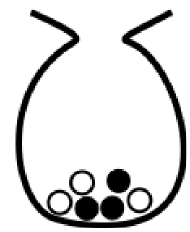
Some students think of the event being modeled as **picking two balls simultaneously**. In that case, the sample space diagram (with labels first selection, second selection) and the probability tree (which again shows a sequence of events) may seem less appropriate. The student may therefore decide to not distinguish between B_1B_2 and B_2B_1. The resulting sample space diagram will be just the upper (or lower) half of the sample space diagram shown above. The resulting probabilities however will remain unaffected.

Lucky Dip

Dominic has devised a simple game.
Inside a bag he places 3 black and 3 white balls. He then shakes the bag.

He asks Amy to take two balls from the bag without looking.





Is Amy right? Is the game fair?

If Amy is wrong, then who is most likely to win?

Show all your reasoning clearly.

Student Sample Responses: Anna

Amy could select
Black + black
Black + white
White + black
White + white

There are 2 when the balls are the same color + 2 when the balls are different

THE GAME IS FAIR

Explain what the student has done.

What isn't clear about her work?

What mistakes has she made?

Student Sample Responses: Ella

		2nd ball					
		B1	B2	B3	W1	W2	W3
1st ball	B1	Amy	A	A	D	D	D
	B2	A	A	A	D	D	D
	B3	A	A	A	D	D	D
	W1	D	D	D	A	A	A
	W2	D	D	D	A	A	A
	W3	D	D	D	A	A	A.

There are 36 equally likely outcomes.

Amy wins 18 times

Dominic wins 18 times

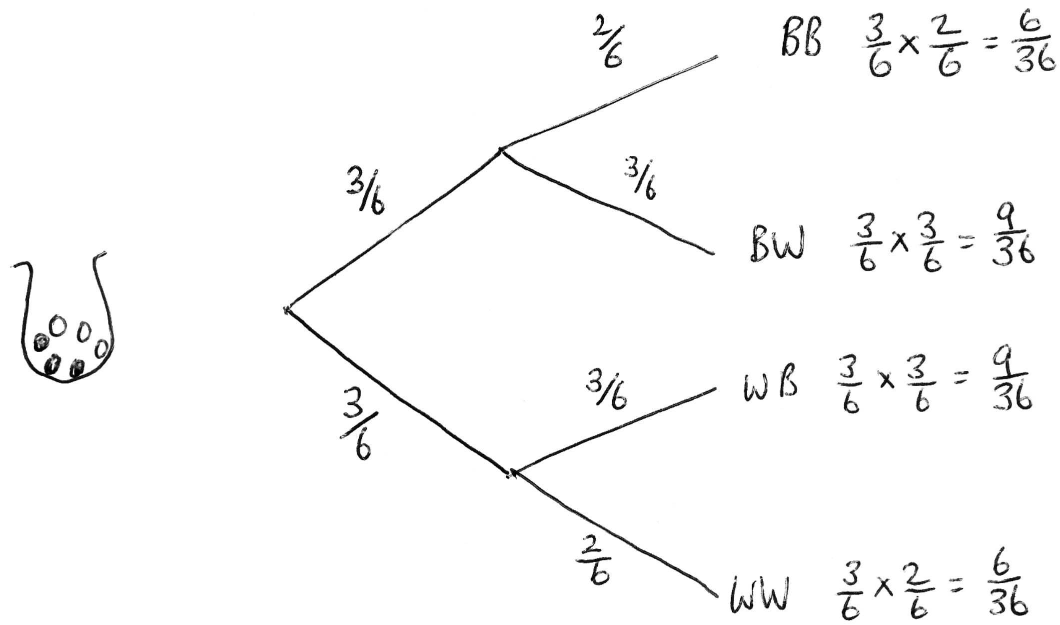
So the game is fair.

Explain what the student has done.

What isn't clear about her work?

What mistakes has she made?

Student Sample Responses: Jordan



2 same color $= \frac{6}{36} + \frac{6}{36} = \frac{12}{36} = \frac{1}{3}$

2 different color $= \frac{9}{36} + \frac{9}{36} = \frac{18}{36} = \frac{1}{2}$

Explain what the student has done. ....

....

....

....

What isn't clear about his work? ....

....

....

....

What mistakes has he made? ....

....

....

....

How Did You Work?

Mark the boxes and complete the sentences that apply to your work.

1. Our group work was better than my own work ☐

   Our group solution was better because

2. Our solution is similar to one of the sample responses ☐

   Our solution is similar to *(add name of the student)*

   I prefer **our solution** / **the student's solution** (*circle)*

   because

OR

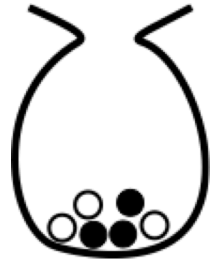
Our solution is different from **all** of the sample responses ☐

Our solution is different from all of the sample responses because

3. What advice would you give a student new to this task about potential pitfalls?

Lucky Dip

Dominic has devised a simple game.



Inside a bag he places 3 black and 3 white balls.
He then shakes the bag.

He asks Amy to take two balls from the bag without looking.

Dominic

If the two balls are the same color, then you win.

If they are different colors, then I win.

Amy

OK, that sounds fair to me.

Is Amy right? Is the game fair?
If Amy is wrong, then who is most likely to win?
Show all your reasoning clearly.

Planning a Joint Solution

1. Take turns to explain how you did the task and how you now think it could be improved.

2. Listen carefully to explanations.
   – Ask questions if you don't understand.

   – Discuss with your partners:
     - What you like/dislike about your partners' math.
     - Any assumptions your partner has made.
     - How their work could be improved.

3. Once everyone in the group has explained their solution, plan a joint approach that is better than each of the separate solutions.
   – On the second side of your poster or paper write a couple of sentences outlining your plan.

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2. Take turns to work through a students' solution.
   - Write your answers on your mini-whiteboards.

3. Explain your answer to the rest of the group.

4. Listen carefully to explanations.
   - Ask questions if you don't understand.

5. Once everyone is satisfied with the explanations, write the answers below the students' solution.
   - Make sure the student who writes the answers is not the student who explained them.

Anna's Response

Amy could select
Black + black
Black + white
White + black
White + white

There are 2 when the balls are the same color + 2 when the balls are different

THE GAME IS FAIR

Ella's Response

2nd ball

1st ball	B1	B2	B3	W1	W2	W3
B1	Amy	A	A	D	D	D
B2	A	A	A	D	D	D
B3	A	A	A	D	D	D
W1	D	D	D	A	A	A
W2	D	D	D	A	A	A
W3	D	D	D	A	A	A.

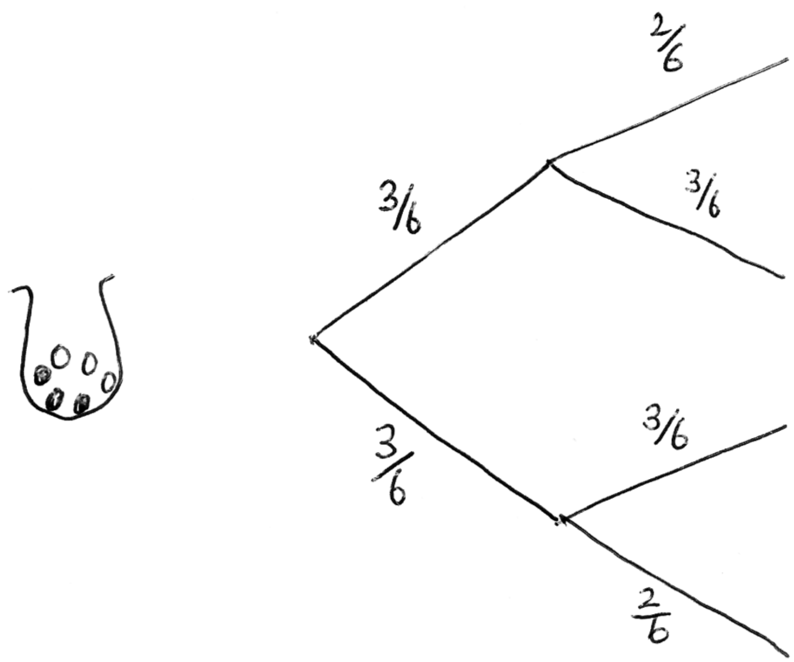
There are 36 equally likely outcomes.

Amy wins 18 times

Dominic wins 18 times

So the game is fair.

Jordan's Response



BB $\frac{3}{6} \times \frac{2}{6} = \frac{6}{36}$

BW $\frac{3}{6} \times \frac{3}{6} = \frac{9}{36}$

WB $\frac{3}{6} \times \frac{3}{6} = \frac{9}{36}$

WW $\frac{3}{6} \times \frac{2}{6} = \frac{6}{36}$

2 same color $= \frac{6}{36} + \frac{6}{36} = \frac{12}{36} = \frac{1}{3}$

2 different color $= \frac{9}{36} + \frac{9}{36} = \frac{18}{36} = \frac{1}{2}$

Mathematics Assessment Project

CLASSROOM CHALLENGES

This lesson was designed and developed by the
Shell Center Team
at the
University of Nottingham
Malcolm Swan, Nichola Clarke, Clare Dawson, Sheila Evans
with
Hugh Burkhardt, Rita Crust, Andy Noyes, and Daniel Pead

It was refined on the basis of reports from teams of observers led by
David Foster, Mary Bouck, and Diane Schaefer
based on their observation of trials in US classrooms
along with comments from teachers and other users.

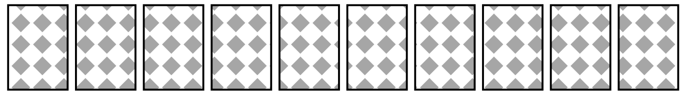
This project was conceived and directed for
MARS: Mathematics Assessment Resource Service
by
Alan Schoenfeld, Hugh Burkhardt, Daniel Pead, and Malcolm Swan
and based at the University of California, Berkeley

We are grateful to the many teachers, in the UK and the US, who trialed earlier versions
of these materials in their classrooms, to their students, and to
Judith Mills, Carol Hill, and Alvaro Villanueva who contributed to the design.

This development would not have been possible without the support of
Bill & Melinda Gates Foundation
We are particularly grateful to
Carina Wong, Melissa Chabran, and Jamie McKee

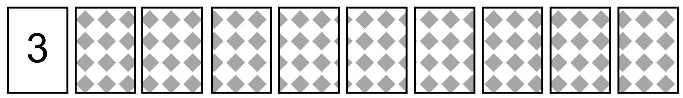
Card Game

Mrs Jakeman is teaching her class about probability.
She has ten cards, numbered 1 to 10, which she mixes up and stands on a shelf turned so that the numbers do not show.



Mrs Jakeman turns the cards round on at a time.
The class has to guess whether the next card will have a higher number than the last one turned or a lower number.

The first card turned is the 3.



1. Would you expect the next number to be higher than 3 or lower?

………………

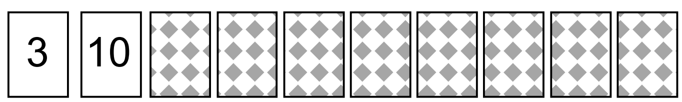
Explain why you made this decision.

……………………………………………………………………………………..

……………………………………………………………………………………..

……………………………………………………………………………………..

The next card is number 10.



2. What is the probability that the next card will be a higher number?

………………

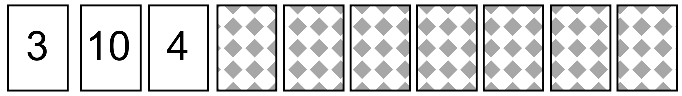
Explain how you know.

……………………………………………………………………………………..

……………………………………………………………………………………..

……………………………………………………………………………………..

The next card is number 4.

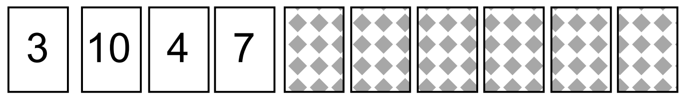


3. What is the probability that the next number is higher?

………………

Show your work.

The next card is number 7.

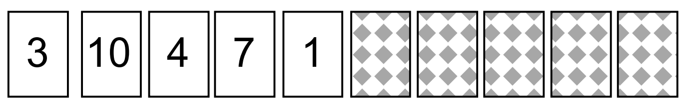


4. What is the probability that the next number is lower?

………………

Show your work.

The fifth card is the 1.



When the sixth card is turned the probability that the following one is higher is the same as the probability that it is lower.

5. What must the sixth card be?

………………

Explain how you worked it out.

…………………………………………………………………………………………..

…………………………………………………………………………………………..