PROBLEM SOLVING

Mathematics Assessment Project

CLASSROOM CHALLENGES

A Formative Assessment Lesson

Geometry Problems: *Circles and Triangles*

Mathematics Assessment Resource Service
University of Nottingham & UC Berkeley
Beta Version

Geometry Problems: Circles and Triangles

MATHEMATICAL GOALS

This lesson unit is intended to help you assess how well students are able to use geometric properties to solve problems. In particular, the lesson will help you identify and help students who have the following difficulties:

- Solving problems by determining the lengths of the sides in right triangles.
- Finding the measurements of shapes by decomposing complex shapes into simpler ones.

The lesson unit will also help students to recognize that there may be different approaches to geometrical problems, and to understand the relative strengths and weaknesses of those approaches.

COMMON CORE STATE STANDARDS

This lesson relates to the following *Mathematical Practices* in the *Common Core State Standards for Mathematics*:

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
6. Attend to precision.
7. Look for and make use of structure.

This lesson gives students the opportunity to apply their knowledge of the following *Standards for Mathematical Content* in the *Common Core State Standards for Mathematics:*

G-SRT: Understand similarity in terms of similarity transformations.
Define trigonometric ratios and solve problems involving right triangles.
G-C: Understand and apply theorems about circles.

INTRODUCTION

- Before the lesson, students attempt the problem individually. You then review their work and create questions for students to answer in order to improve their solutions.
- During the lesson, students work collaboratively in small groups to produce an improved solution to the same problem.
- Working in the same small groups, students comment on and evaluate some solutions produced by students in another class.
- In a whole-class discussion, students explain and compare the alternative solution strategies they have seen and used.
- Finally, students review the work they did on their individual solutions and write about what they learned.

MATERIALS REQUIRED

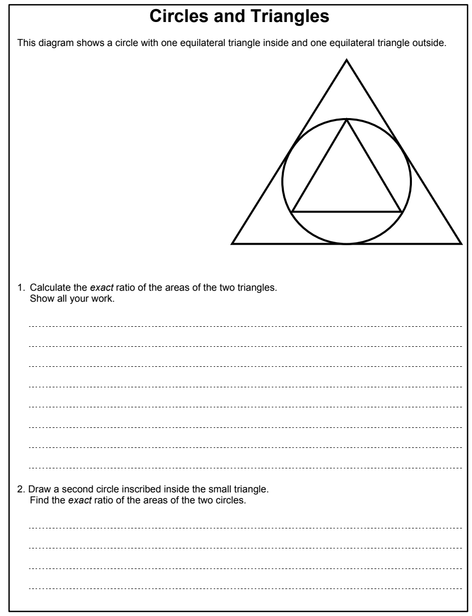
- Each individual student will need a copy of the task sheet *Circles and Triangles*, a ruler, calculator, pencil, mini-whiteboard, pen and eraser.
- Each small group of students will need a copy of *Sample Responses to Discuss,* and a large sheet of paper for making a poster.
- There are also some slides to help with instructions and to support whole-class discussion.

TIME NEEDED

15minutes before the lesson, a 1-hour lesson, and 10 minutes in a follow-up lesson (or for homework). Timings are approximate and will depend on the needs of the class.

BEFORE THE MAIN LESSON

Assessment task: *Circles and Triangles* (15 minutes)


Circles and Triangles

This diagram shows a circle with one equilateral triangle inside and one equilateral triangle outside.

1. Calculate the *exact* ratio of the areas of the two triangles.
   Show all your work.

2. Draw a second circle inscribed inside the small triangle.
   Find the *exact* ratio of the areas of the two circles.

Have students do this task in class or for homework a day or more before the formative assessment lesson. This will give you an opportunity to assess the work and to find out the kinds of difficulties students have with it. Then you will be able to target your help more effectively in the follow-up lesson.

Give out the task *Circles and Triangles*, a pencil, and a ruler. Issue calculators if students ask for them.

Introduce the task briefly and help the class to understand the problem and its context.

> *Read through the task and try to answer it as carefully as you can.*
>
> *Show all your work so that I can understand your reasoning.*
>
> *Don't worry too much if you don't understand everything, because there will be a lesson [tomorrow] using this task.*

It is important that students are allowed to answer the questions without assistance, as far as possible. If students are struggling to get started then ask questions that help them understand what is required, but make sure you do not do the task for them.

Students who sit together often produce similar answers, and then when they come to compare their work, they have little to discuss. For this reason we suggest that, when students do the task individually, you ask them to move to different seats. Then at the beginning of the formative assessment lesson, allow them to return to their usual seats. Experience has shown that this produces more profitable discussions.

When all students have made a reasonable attempt at the task, tell them that they will have time to revisit and revise their solutions later.

Assessing students' responses

We suggest that you do not write scores on students' work. The research shows that this is counterproductive, as it encourages students to compare scores, and distracts their attention from what they might do to improve their mathematical work.

Instead, help students to make further progress by summarizing their difficulties as a series of questions. Some suggestions for these are given on the next page. These have been drawn from common difficulties observed in trials of this lesson unit.

We suggest that you write your own lists of questions, based on your own students' work, using these ideas. You may choose to write questions on each student's work. If you do not have time to do this, select a few questions that will help the majority of students. These can then be written on the board at the beginning of the lesson.

Common issues:	**Suggested questions and prompts:**
Student has difficulty getting started	• What do you know about the angles or lines in the diagram? How can you use what you know? What do you need to find out? • You may find it helps to give a name to some of the lengths. Try r for the radius of the circle, x for the side of the big triangle, and so on. • Can you add any helpful construction lines to your diagram? What do you know about these lines? • Can you find relationships between the lengths from what you know about geometry?
Student works out the ratio by measuring the dimensions of the triangles	• What are the advantages/disadvantages of your method? • Are your measurements accurate enough? How do you know?
Student does not explain the method clearly **For example: The student does not explain why triangles are similar.** Or: The student does not explain why triangles are congruent.	• Would someone unfamiliar with your type of solution easily understand your work? • How do you know these triangles are similar/congruent? • It may help to label points and lengths in the diagram.
Student has problems recalling standard ratios The student recalls incorrectly or makes an error using the special ratios for a 30°, 60°, 90° triangle $(1, \sqrt{3}, 2)$.	• What do you know about cos 30°? What do you know about sin 30°? How can you use this information? • Use the Pythagorean Theorem to check/calculate the ratio of the sides of the triangle.
Student uses perception alone to calculate the ratio For example: The student rotates the small triangle about the center of the circle and assumes that the diagram alone is enough to show the ratio of areas is 4:1.	• What math can you use to justify your answer?
Student makes a technical error For example: The student makes an error manipulating an equation.	• Check to see if you have made any algebraic errors.

Common issues:	**Suggested questions and prompts:**
Student uses ratios of lengths rather than ratios of areas For example: When finding the ratio of the areas of the two circles, the student obtains an incorrect answer because they find the ratio of the radii, rather than the ratio of the squares of the radii.	• What is the formula for the area of the circle? How can you use it to find the ratio of the areas of the circles?
Student produces correct solutions	• Can you solve the problem using a different method? Which method do you prefer? Why?

SUGGESTED LESSON OUTLINE

Improve individual solutions to *Circles and Triangles* (10 minutes)

Return students' papers and give each student a mini-whiteboard, pen and eraser.

Recall what we were looking at in a previous lesson. What was the task?

I have read your solutions and I have some questions about your work.

If you have not added questions to individual pieces of work, write your list of questions on the board, and ask students to select questions appropriate to their own work.

Ask students to spend a few minutes answering your questions. It is helpful if they do this using mini-whiteboards, so that you can see what they are writing.

I would like you to work on your own to answer my questions for about ten minutes.

Collaborative small-group work on *Circles and Triangles* (10 minutes)

When students have made a reasonable attempt at the task on their own, organize them into groups of two or three. Give each group a large, fresh piece of paper and a felt-tipped pen. Ask students to have another go at the task, but this time ask them to combine their ideas and make a poster to show their solutions.

Put your own work aside until later in the lesson. I want you to work in groups now.

Your task is to work together to produce a solution that is better than your individual solutions.

While students work in small groups you have two tasks, to note their different approaches to the task, and to support their reasoning.

Note different student approaches to the task

What mathematics do students choose to use? Have they moved on from the mathematical choices made in the assessment task? Do they measure the lengths of the sides of the triangles? Do they draw construction lines? Do they use similar triangles? Do they use algebra? Do they use proportion?

Do students attempt to use the special ratios for 30°, 60°, 90° triangles ($1 : \sqrt{3} : 2$)? If so, how do they do this?

When finding the ratio of the areas of the two triangles, do they find the ratio of the squares of the bases or do they use an alternative method? When finding the ratio of the areas of the two circles, do students find the ratio of the squares of the radii or do they use an alternative method?

Do students fully explain their solutions?

Note any errors, and think about your understanding of students' strengths and weaknesses from the assessment task. You can use this information to focus whole-class discussion towards the end of the lesson.

Support student problem solving

Try not to make suggestions that move students towards a particular approach to the task. Instead, ask questions that help students to clarify their thinking. Focus on supporting students' strategies rather than finding the numerical solution. You may find the questions on the previous page helpful.

If the whole class is struggling on the same issue, write relevant questions on the board.

You may find that some students think the empirical approach (measuring the diagram) is best.

Will your answer change if you measure in inches rather than millimeters?

This question may focus students' attention on the lack of units of measure in the solution and the problem of accuracy.

What are the strengths/weaknesses of this approach?

Are your measurements exact?

Do you think that, if we asked another group that used this same method, they would come up with exactly the same answer as you?

Collaborative small-group analysis of *Sample Responses to Discuss* (20 minutes)

When students have had sufficient time to attempt the problem in their group, give each group copies of the *Sample Responses to Discuss*. This task gives students an opportunity to evaluate a variety of possible approaches to the task, without providing a complete solution strategy.

You may decide there is not enough time for each group to work through all four pieces of work. In that case, be selective about what you hand out. For example, groups that have successfully completed the task using one method will benefit from looking at different approaches. Other groups, that have struggled with a particular approach may benefit from seeing a student version of the same strategy.

Here are some different solutions to the problem.

Compare these solutions with your own.

Imagine you are the teacher. Describe how the student approached the problem.

Write your explanation on each solution.

What do you like/dislike about the work?

What isn't clear about the work?

What questions would you like to ask this student?

To encourage students to do more than check to see if the answer is correct, you may wish to use the projector resource *Analyzing Sample Responses to Discuss*. During the group work, check to see which of the explanations students find more difficult to understand.

Plenary whole-class discussion: comparing different solution methods (15 minutes)

Organize a whole-class discussion comparing the four given solutions. Collect comments and ask for explanations.

We are going to look at and compare the four solutions.

Can you explain Bill's method?

Why does Carla draw another triangle in the inner circle?

Encourage students to challenge explanations while keeping your own interventions to a minimum.

Do you agree with Tyler's explanation? [If yes] Explain again, in your own words. [If no] Explain what you think, then.

Finally, ask students to evaluate and compare methods.

Which one did you like best? Why?

Which approach did you find most difficult to understand? Why?

Did anyone come up with a method different from these?

Some issues that might be discussed, with suggested questions and prompts, are given below.

Anya uses measurement

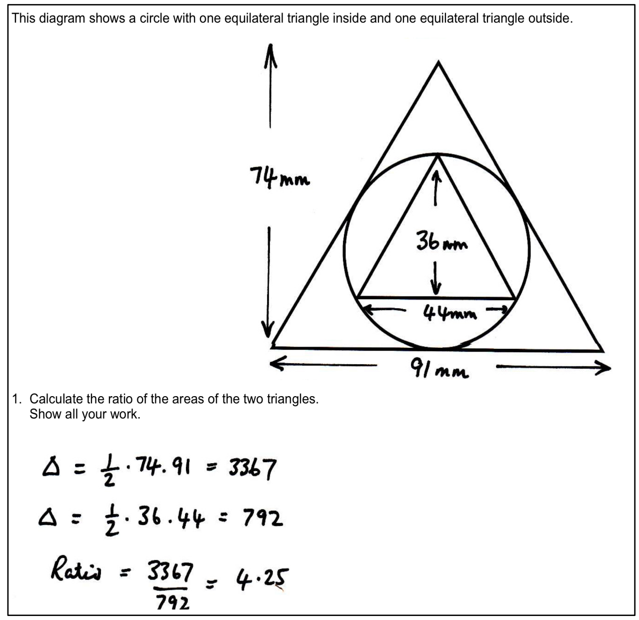
Strengths: It is easy to do. It gives you a feeling for the answer. Anya's calculations are correct. She has rounded to two decimal places.

Weaknesses: You only know it is true for the particular case you measure. It's not exact. It doesn't tell you why it's true. Anya does not calculate the areas of the circles, or their ratio (About: $25^2 : 11^2 = 5 : 1$).

Do you think Anya's answer is accurate?

Would an answer rounded to four decimal places be better?

What do you think the answer should be?

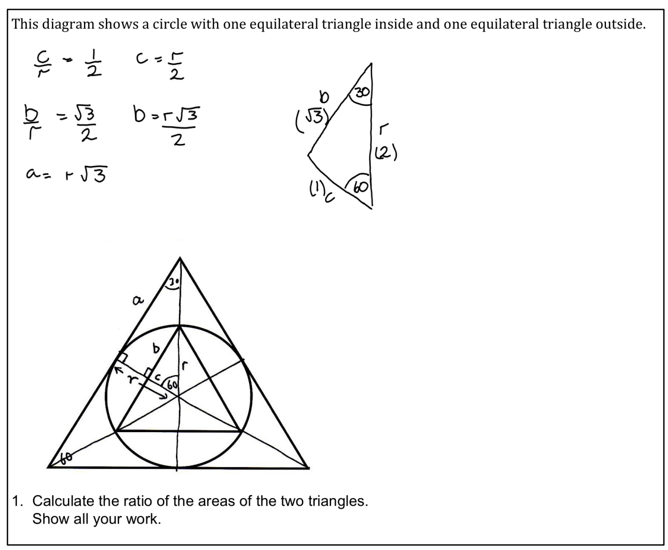


Bill uses algebra and ratios

Strengths: Bill's method does not depend on the size of the diagram.
You can use this method for all sorts of problems.

Weaknesses: Bill's work is difficult to follow. There are gaps in his explanation, and it is quite difficult work. Bill does not answer the question, as he does not calculate the ratio of the areas of the triangles.

He does not explain why the side lengths in the triangle are in the ratios he writes down, which is based on these trigonometric ratios:



$$\sin 30° = \frac{c}{r} = \frac{1}{2}$$

$$\cos 30° = \frac{b}{r} = \frac{\sqrt{3}}{2}$$

$$\tan 60° = \frac{a}{r} = \sqrt{3}$$

You could ask students to explain where the ratios in Bill's solution come from, and then to use the lengths to complete the solution.

Why does $\frac{c}{r} = \frac{1}{2}$?

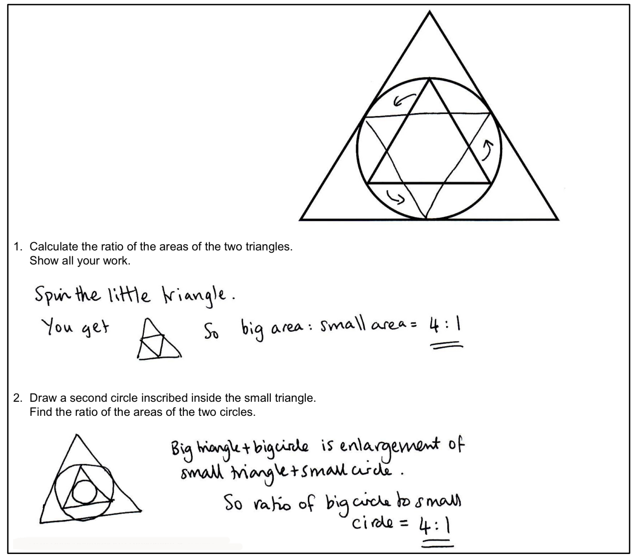
Why does $\frac{b}{r} = \frac{\sqrt{3}}{2}$?

Why does $a = r\sqrt{3}$?

Why does Bill multiply by 6?

What is the ratio of the areas of the two triangles?

Carla uses transformations – rotation and enlargement



1. Calculate the ratio of the areas of the two triangles.
   Show all your work.

Spin the little triangle.

You get [drawing] So big area : small area = 4 : 1

2. Draw a second circle inscribed inside the small triangle.
   Find the ratio of the areas of the two circles.

Big triangle + big circle is enlargement of small triangle + small circle.

So ratio of big circle to small circle = 4 : 1

Strengths: It is simple. It is clear, even elegant. It is easy to do.

Weaknesses: You have to see it! There are some gaps in the explanation that need to be completed.

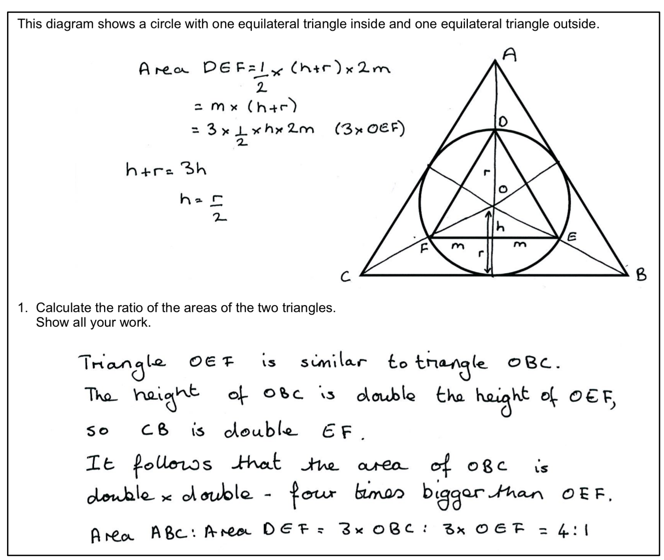
How do you know that, if you rotate the small triangle, it hits the midpoints of the large triangle?

How do you know the four small triangles are congruent?

How do you know the four small triangles are equilateral?

How do you know the circle has been enlarged in the same ratio as the triangle?

Darren uses algebra and similar triangles



This diagram shows a circle with one equilateral triangle inside and one equilateral triangle outside.

Area DEF $= \frac{1}{2} \times (h+r) \times 2m$

$= m \times (h+r)$

$= 3 \times \frac{1}{2} \times h \times 2m$ $(3 \times OEF)$

$h + r = 3h$

$h = \frac{r}{2}$

1. Calculate the ratio of the areas of the two triangles.
   Show all your work.

Triangle OEF is similar to triangle OBC.
The height of OBC is double the height of OEF,
so CB is double EF.
It follows that the area of OBC is
double × double - four times bigger than OEF.
Area ABC : Area DEF = 3 × OBC : 3 × OEF = 4 : 1

Strengths: Darren's method does not depend on the size of the diagram. Darren has labeled the diagram: this makes his work easier to understand.

Weaknesses: Darren's work is difficult to follow at times. He has failed to explain part of his work.

Are triangles OBC and OEF similar? How do you know?

What does Darren mean by "double × double"?

Can you use math to show Darren's answer is correct?

Through comparing different methods, students may come to realize the power of using different methods to solve the same problem.

Next lesson: review individual solutions to *Circles and Triangles* (10 minutes)

Ask students to look again at their original individual solutions to the problem.

Read through your original solution to the Circles and Triangles problem.

Write what you have learned during the lesson.

Suppose a friend began work on this task tomorrow. What advice would you give your friend to help him or her produce a good solution?

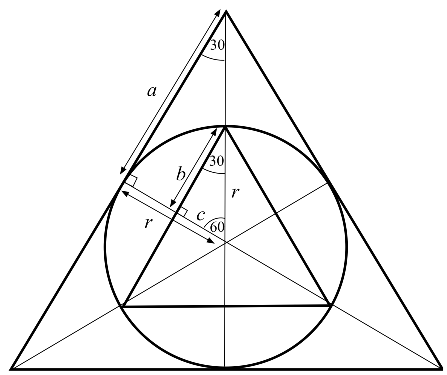
Some teachers set this task as homework.

SOLUTIONS

Circles and Triangles

Bill's method:

Question 1



$$\cos 30° = \frac{b}{r} = \frac{\sqrt{3}}{2} \qquad \Rightarrow b = \frac{r\sqrt{3}}{2}$$

$$\sin 30° = \frac{c}{r} = \frac{1}{2} \qquad \Rightarrow c = \frac{r}{2}$$

$$\tan 60° = \frac{a}{r} = \sqrt{3} \qquad \Rightarrow a = r\sqrt{3}$$

Area of small equilateral triangle:

$$6 \times \frac{1}{2} \times b \times c = 3 \times \frac{r\sqrt{3}}{2} \times \frac{r}{2} = \frac{3\sqrt{3}r^2}{4}$$

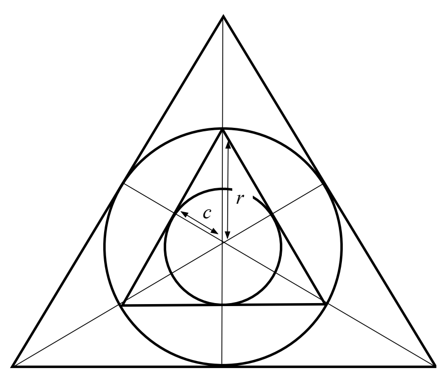
Area of large equilateral triangle:

$$6 \times \frac{1}{2} \times a \times r = 3 \times r\sqrt{3} \times r = 3\sqrt{3}r^2$$

Ratio of area of the outer to the area of the inner triangle =

$$3\sqrt{3}r^2 : \frac{3\sqrt{3}}{4}r^2 = 4:1.$$

Question 2



c is the radius of the inscribed circle. $c = \frac{r}{2}$

Ratio of area of circles:

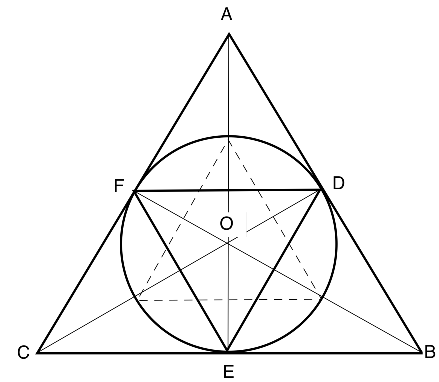
$$= \pi r^2 : \pi c^2$$

$$= \pi r^2 : \pi \frac{r^2}{4}$$

$$= 4:1.$$

Carla's method

Question 1



The small equilateral triangle is rotated through 60° about O, the center of the circle. The arm of the rotation is the radius of the circle. Therefore points D, E, and F are all points on the circumference of the circle.

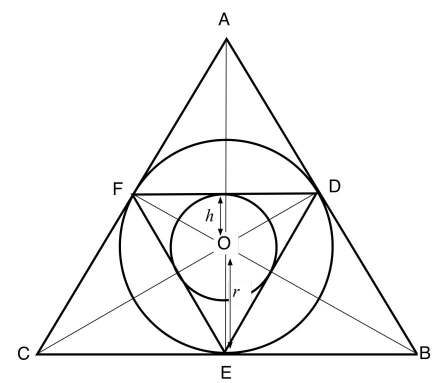
These points bisect the sides of Δ ABC.

Δ CFE is isosceles (CF = CE because the lengths of two tangents to a circle from a point are equal), so ∠CFE = ∠FEC = (180 – 60) ÷ 2 = 60°.
Therefore Δ CFE is equilateral.

It follows by symmetry that all four small triangles are equilateral and congruent.

Hence the ratio of the area of the outer to the area of the inner triangle = 4 : 1.

Question 2



Ratio of the area of the outer to the area of the inner triangle:

= (3 × area Δ OCB) : (3 × area Δ FDO)

$$= \left(3 \times \frac{1}{2} \times r \times CB\right) : \left(3 \times \frac{1}{2} \times h \times \frac{1}{2} \times CB\right)$$

$$= 2r \; : \; h.$$

Since we know from Q1 that this ratio is 4 : 1 $\Rightarrow h = \dfrac{r}{2}$.

Ratio of area of circles

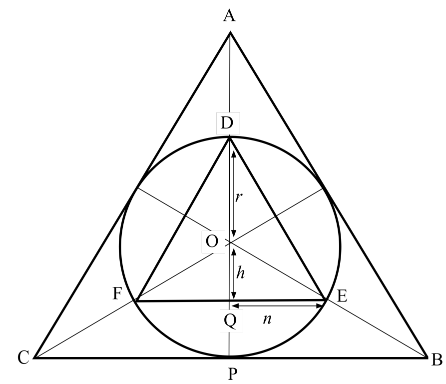
$$= \pi r^2 \; : \; \pi h^2$$

$$= \pi r^2 : \pi \frac{r^2}{4}$$

$$= 4 \; : \; 1.$$

Darren's method

Question 1



Area of Δ DEF = 3 × area of Δ OEF

$$\Rightarrow \frac{1}{2} \times 2n \times (h + r) = 3 \times \frac{1}{2} \times 2n \times h$$

$$\Rightarrow n(h + r) = 3nh$$

$$\Rightarrow h + r = 3h$$

$$\Rightarrow h = \frac{r}{2}$$

Triangle OQE is similar to triangle OPB:

∠POB is common to both triangles and OQE = OPB = 90° (altitudes of an equilateral triangle).

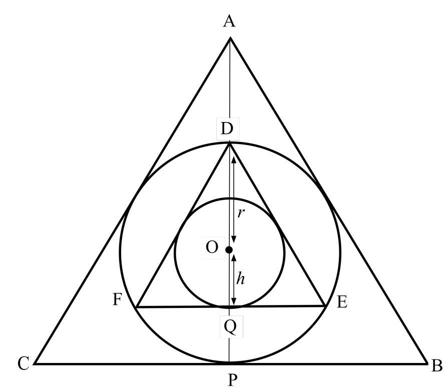
Therefore PB = $2n$ and so CB = $4n$ (altitudes of an equilateral triangle bisect a side).

Area Δ ABC = 3 x area Δ OBC $= 3 \times \frac{1}{2} \times 4n \times r = 6nr$.

Area Δ DEF = 3 x area Δ OEF $= 3 \times \frac{1}{2} \times 2n \times \frac{r}{2} = \frac{3nr}{2}$.

Ratio of areas of triangles $= 6nr : \frac{3nr}{2} = 4:1$.

Question 2



h is the radius of the inscribed circle.

$h = \frac{r}{2}$.

The ratio of the area of the outer circle to the area of the inner circle:

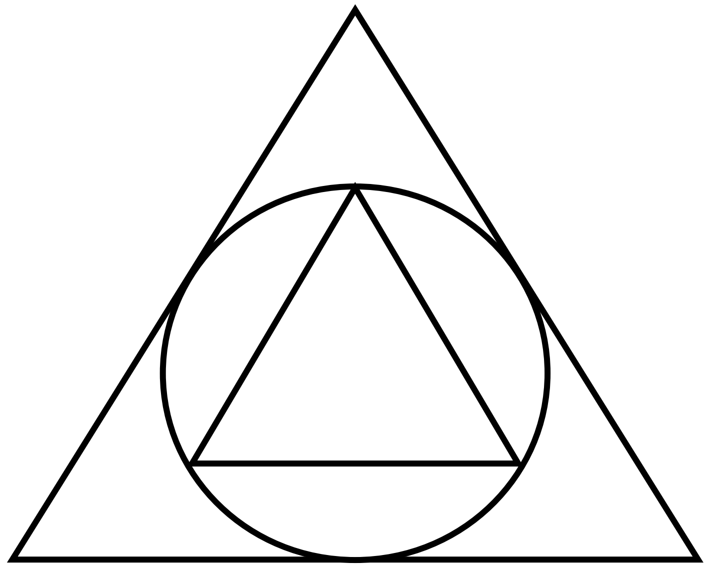
$= \pi r^2 \ : \ \pi h^2$

$= \pi r^2 \ : \ \pi \frac{r^2}{4}$

$= 4 \ : \ 1$.

Circles and Triangles

This diagram shows a circle with one equilateral triangle inside and one equilateral triangle outside.



1. Calculate the *exact* ratio of the areas of the two triangles.
   Show all your work.

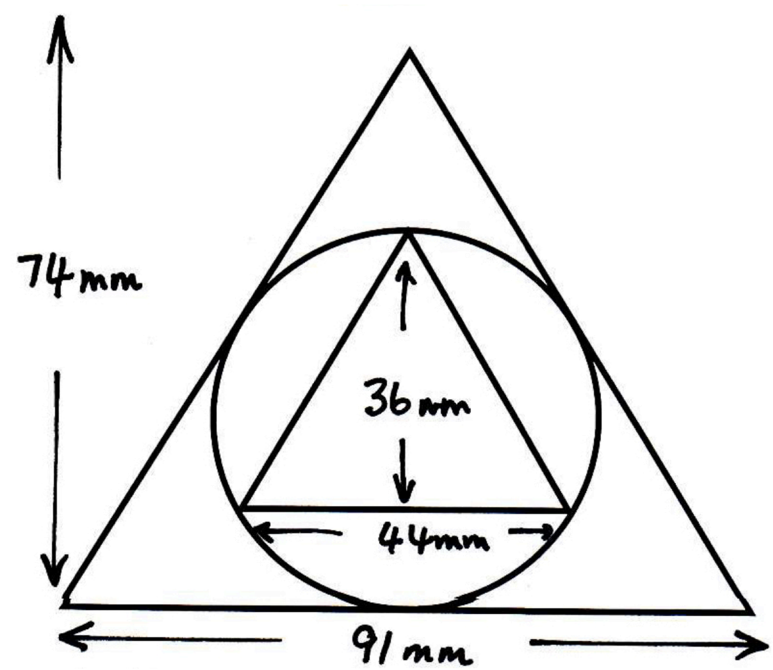
2. Draw a second circle inscribed inside the small triangle.
   Find the *exact* ratio of the areas of the two circles.

Sample Responses to Discuss: Anya

Imagine you are Anya's teacher. Describe how Anya approached the problem.

Write your explanation on a separate sheet.
What do you like/dislike about the work?
What is unclear about the work?
What questions would you like to ask Anya?

This diagram shows a circle with one equilateral triangle inside and one equilateral triangle outside.



1. Calculate the ratio of the areas of the two triangles.
   Show all your work.

$$\Delta = \frac{1}{2} \cdot 74 \cdot 91 = 3367$$

$$\Delta = \frac{1}{2} \cdot 36 \cdot 44 = 792$$

$$\text{Ratio} = \frac{3367}{792} = 4.25$$

Sample Responses to Discuss: Bill

Imagine you are Bill's teacher. Describe how Bill approached the problem.

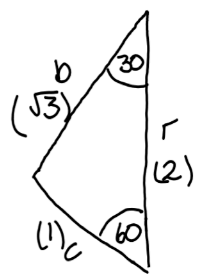
Write your explanation on a separate sheet.
What do you like/dislike about the work?
What is unclear about the work?
What questions would you like to ask Bill?

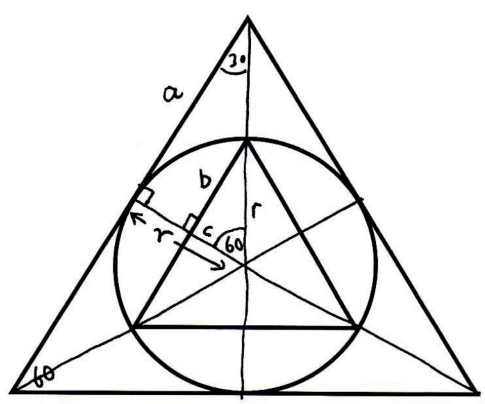
This diagram shows a circle with one equilateral triangle inside and one equilateral triangle outside.

$\frac{c}{r} = \frac{1}{2}$ $c = \frac{r}{2}$

$\frac{b}{r} = \frac{\sqrt{3}}{2}$ $b = \frac{r\sqrt{3}}{2}$

$a = r\sqrt{3}$





1. Calculate the ratio of the areas of the two triangles.
   Show all your work.

Area of small triangle $= 6 \times \frac{1}{2} \times b \times c = 6 \times \frac{1}{2} \times \frac{r\sqrt{3}}{2} \times \frac{r}{2} = \frac{3r^2\sqrt{3}}{4}$

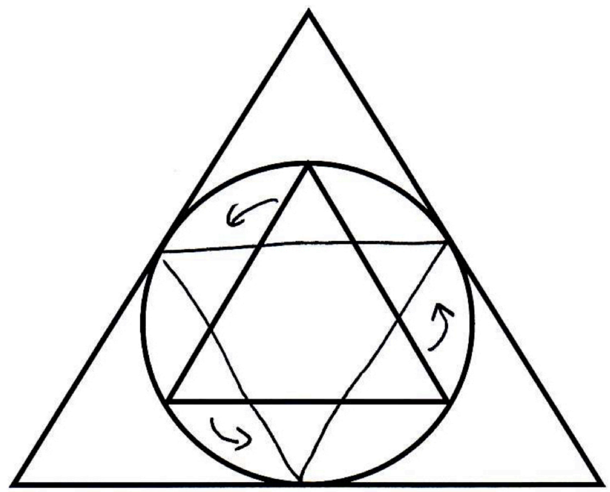
Area of large triangle $= 6 \times \frac{1}{2} \times a \times r = 3 \times r\sqrt{3} \times r = 3r^2\sqrt{3}$

Sample Responses to Discuss: Carla

Imagine you are Carla's teacher. Describe how Carla approached the problem.

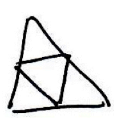
Write your explanation on a separate sheet.
What do you like/dislike about the work?
What isn't clear about the work?
What questions would you like to ask Carla?

This diagram shows a circle with one equilateral triangle inside and one equilateral triangle outside.

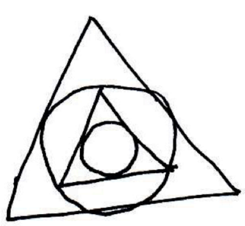


1. Calculate the ratio of the areas of the two triangles.
   Show all your work.

Spin the little triangle.

You get  So big area : small area = 4 : 1

2. Draw a second circle inscribed inside the small triangle.
   Find the ratio of the areas of the two circles.



Big triangle + big circle is enlargement of small triangle + small circle.

So ratio of big circle to small circle = 4 : 1

Sample Responses to Discuss: Darren

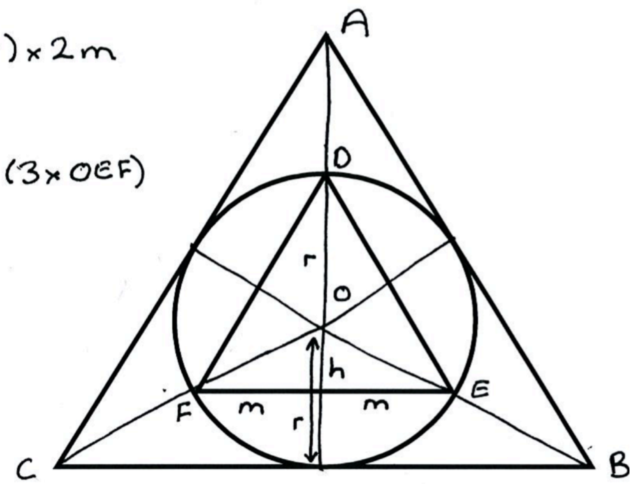
Imagine you are Darren's teacher. Describe how Darren approached the problem.

Write your explanation on a separate sheet.
What do you like/dislike about the work?
What is unclear about the work?
What questions would you like to ask Darren?

This diagram shows a circle with one equilateral triangle inside and one equilateral triangle outside.

$$\text{Area } DEF = \frac{1}{2} \times (h+r) \times 2m$$
$$= m \times (h+r)$$
$$= 3 \times \frac{1}{2} \times h \times 2m \quad (3 \times OEF)$$

$$h + r = 3h$$
$$h = \frac{r}{2}$$



1. Calculate the ratio of the areas of the two triangles.
   Show all your work.

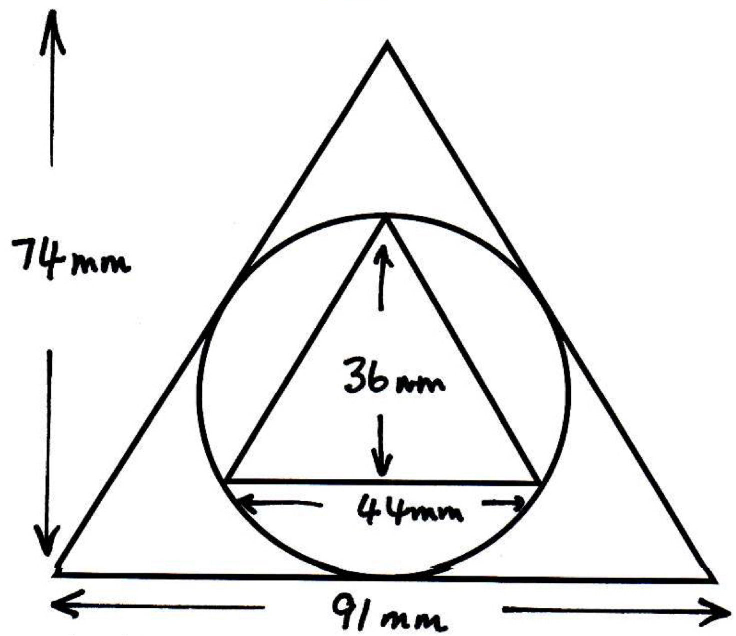
Triangle OEF is similar to triangle OBC.
The height of OBC is double the height of OEF,
so CB is double EF.
It follows that the area of OBC is
double × double - four times bigger than OEF.
Area ABC : Area DEF = 3 × OBC : 3 × OEF = 4 : 1

Analyzing Sample Responses to Discuss

- Explain what the student has done.
- What do you like/dislike about the work?
- What is unclear about the work?
- What questions would you like to ask this student?

Anya's Solution

This diagram shows a circle with one equilateral triangle inside and one equilateral triangle outside.



1. Calculate the ratio of the areas of the two triangles.
   Show all your work.

$$\Delta = \frac{1}{2} \cdot 74 \cdot 91 = 3367$$

$$\Delta = \frac{1}{2} \cdot 36 \cdot 44 = 792$$

$$\text{Ratio} = \frac{3367}{792} = 4 \cdot 25$$

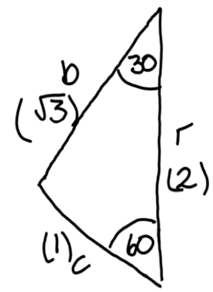
Bill's Solution (1)

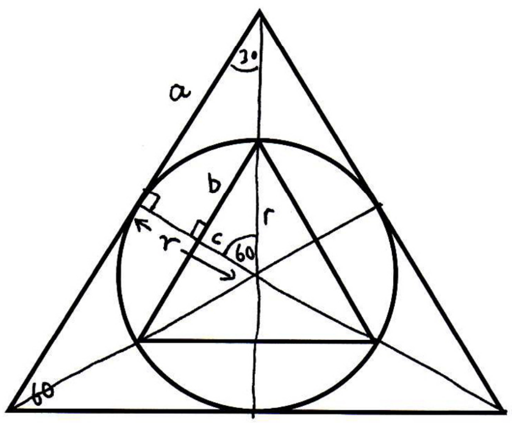
This diagram shows a circle with one equilateral triangle inside and one equilateral triangle outside.

$\frac{c}{r} = \frac{1}{2}$ $\quad c = \frac{r}{2}$

$\frac{b}{r} = \frac{\sqrt{3}}{2}$ $\quad b = \frac{r\sqrt{3}}{2}$

$a = r\sqrt{3}$





1. Calculate the ratio of the areas of the two triangles.
   Show all your work.

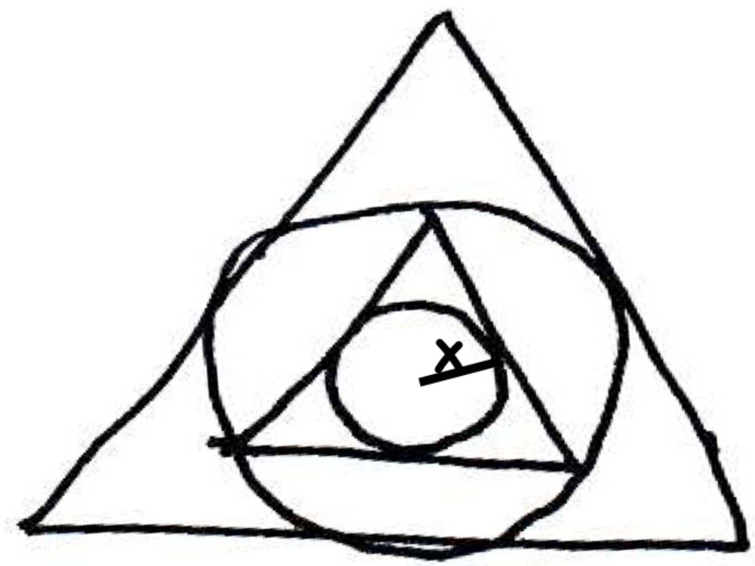
Bill's Solution (2)

2. Draw a second circle inscribed inside the small triangle.
   Find the ratio of the areas of the two circles.

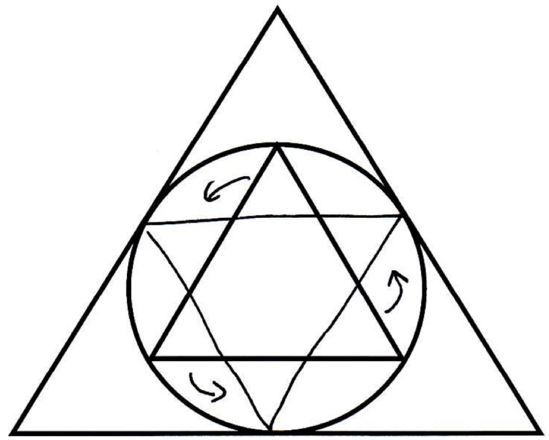
Large circle $= \pi r^2$

Small circle $= \pi x^2 = \pi \left(\frac{r}{2}\right)^2 = \frac{\pi r^2}{4}$

Ratio of areas $= 1 : \frac{1}{4} = \underline{\underline{4:1}}$

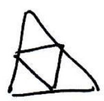


Carla's Solution

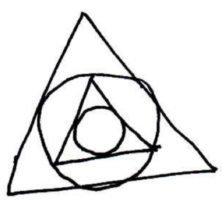


1. Calculate the ratio of the areas of the two triangles.
   Show all your work.

Spin the little triangle.

You get  So big area : small area = 4 : 1

2. Draw a second circle inscribed inside the small triangle.
   Find the ratio of the areas of the two circles.



Big triangle + big circle is enlargement of small triangle + small circle.

So ratio of big circle to small circle = 4 : 1

Darren's Solution

This diagram shows a circle with one equilateral triangle inside and one equilateral triangle outside.

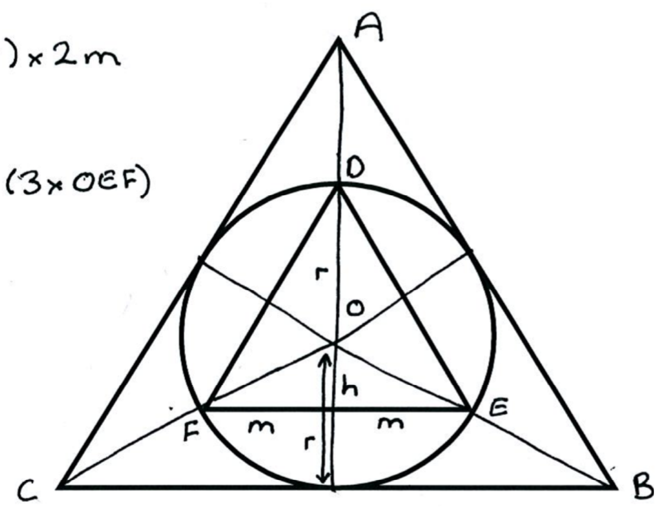
Area DEF $= \frac{1}{2} \times (h+r) \times 2m$

$= m \times (h+r)$

$= 3 \times \frac{1}{2} \times h \times 2m \quad (3 \times OEF)$

$h + r = 3h$

$h = \frac{r}{2}$



1. Calculate the ratio of the areas of the two triangles.
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Triangle OEF is similar to triangle OBC.
The height of OBC is double the height of OEF,
so CB is double EF.
It follows that the area of OBC is
double × double - four times bigger than OEF.
Area ABC : Area DEF = 3 × OBC : 3 × OEF = 4 : 1

Mathematics Assessment Project

CLASSROOM CHALLENGES

This lesson was designed and developed by the
Shell Center Team
at the
University of Nottingham
Malcolm Swan, Nichola Clarke, Clare Dawson, Sheila Evans
with
Hugh Burkhardt, Rita Crust, Andy Noyes, and Daniel Pead

It was refined on the basis of reports from teams of observers led by
David Foster, Mary Bouck, and Diane Schaefer
based on their observation of trials in US classrooms
along with comments from teachers and other users.

This project was conceived and directed for
MARS: Mathematics Assessment Resource Service
by
Alan Schoenfeld, Hugh Burkhardt, Daniel Pead, and Malcolm Swan
and based at the University of California, Berkeley

We are grateful to the many teachers, in the UK and the US, who trialed earlier versions
of these materials in their classrooms, to their students, and to
Judith Mills, Carol Hill, and Alvaro Villanueva who contributed to the design.

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Bill & Melinda Gates Foundation
We are particularly grateful to
Carina Wong, Melissa Chabran, and Jamie McKee

Orchestrating productive mathematical discussions: Five practices for helping teachers move beyond show and tell

The Importance of Discussion

Mathematical discussions are a key part of current visions of effective mathematics teaching

- To encourage student construction of mathematical ideas
- To make student's thinking public so it can be guided in mathematically sound directions
- To learn mathematical discourse practices

Some Sources of the Challenge in Facilitating Discussions

- Lack of familiarity
- Reduces teachers' perceived level of control
- Requires complex, split-second decisions
- Requires flexible, deep, and interconnected knowledge of content, pedagogy, and students

The Five Practices (+)

1. **Anticipating** (e.g., Fernandez & Yoshida, 2004; Schoenfeld, 1998)
2. **Monitoring** (e.g., Hodge & Cobb, 2003; Nelson, 2001; Shifter, 2001)
3. **Selecting** (e.g., Lampert, 2001; Stigler & Hiebert, 1999)
4. **Sequencing** (e.g., Schoenfeld, 1998)
5. **Connecting** (e.g., Ball, 2001; Brendehur & Frykholm, 2000)

Before the Practices: Setting Goals and Selecting Tasks

- **It involves:**
  - Identifying what students are to know and understand about mathematics as a result of their engagement in a particular lesson
  - Being as specific as possible so as to establish a clear target for instruction that can guide the selection of instructional activities and the use of the five practices
- **It is supported by:**
  - Thinking about what students will come to know and understand rather than only on what they will do
  - Consulting resources that can help in unpacking big ideas in mathematics
  - Working in collaboration with other teachers

Selecting a Task

- **It involves:**
  - •Identifying a mathematical task that is aligned with the lesson goals
  - •Making sure the task is rich enough to support a discussion (i.e., a cognitively challenging mathematical task)
- **It is supported by:**
  - •Setting a clear and explicit goal for learning
  - •Using the **Task Analysis Guide** which provides a list of characteristics of tasks at different levels of cognitive demand
  - •Working in collaboration with colleagues

Task Analysis Guide

Lower-Level Demands	Higher-Level Demands
Memorization • involve either reproducing previously learned facts, rules, formulae or definitions OR committing facts, rules, formulae or definitions to memory. • cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure. • are not ambiguous. Such tasks involve exact reproduction of previously-seen material and what is to be reproduced is clearly and directly stated. • have no connection to the concepts or meaning that underlie the facts, rules, formulae or definitions being learned or reproduced.	Procedures With Connections • focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas. • suggest pathways to follow (explicitly or implicitly) that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts. • usually are represented in multiple ways (e.g., visual diagrams, manipulatives, symbols, problem situations). Making connections among multiple representations helps to develop meaning. • require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with the conceptual ideas that underlie the procedures in order to successfully complete the task and develop understanding.
Procedures Without Connections • are algorithmic. Use of the procedure is either specifically called for or its use is evident based on prior instruction, experience, or` placement of the task. • require limited cognitive demand for successful completion. There is little ambiguity about what needs to be done and how to do it. • have no connection to the concepts or meaning that underlie the procedure being used. • are focused on producing correct answers rather than developing mathematical understanding. • require no explanations or explanations that focuses solely on describing the procedure that was used.	Doing Mathematics • require complex and non-algorithmic thinking (i.e., there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example). • require students to explore and understand the nature of mathematical concepts, processes, or relationships. • demand self-monitoring or self-regulation of one's own cognitive processes. • require students to access relevant knowledge and experiences and make appropriate use of them in working through the task. • require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions. • require considerable cognitive effort and may involve some level of anxiety for the student due to the unpredictable nature of the solution process required.

Figure 2. 3 Characteristes of mathematical instructional tasks*.

*These characteristics are derived from the work of Doyle on academic tasks (1988), Resnick on high-level thinking skills (1987), and from the examination and categorization of hundreds of tasks used in QUASAR classrooms (Stein, Grover, & Henningsen, 1996; Stein, Lane, and Silver, 1996).

1. Anticipating likely student responses to mathematical problems

- **It involves considering:**
    - The array of strategies that students might use to approach or solve a challenging mathematical task
    - How to respond to what students produce
    - Which strategies will be most useful in addressing the mathematics to be learned
- **It is supported by:**
    - Doing the problem in as many ways as possible
    - Discussing the problem with other teachers
    - Drawing on relevant research
    - Documenting student responses year to year

2. Monitoring students' actual responses during independent work

- **It involves:**
    - Circulating while students work on the problem and watching and listening
    - Recording interpretations, strategies, and points of confusion
    - Asking questions to get students back "on track" or to advance their understanding
- **It is supported by:**
    - Anticipating student responses beforehand
    - Carefully listening and asking probing questions
    - Using recording tools

Monitoring Tool

Strategy	Who and What	Order
List the different solution paths you anticipated		

3. Selecting student responses to feature during discussion

- **It involves:**
    - Choosing particular students to present because of the mathematics available in their responses
    - Making sure that over time all students are seen as authors of mathematical ideas and have the opportunity to demonstrate competence

        - Gaining some control over the content of the discussion (no more "who wants to present next?")
- **It is supported by:**
    - Anticipating and monitoring
    - Planning in advance which types of responses to select

4. Sequencing student responses during the discussion

- **It involves:**
    - Purposefully ordering presentations so as to make the mathematics accessible to all students
    - Building a mathematically coherent story line
- **It is supported by:**
    - Anticipating, monitoring, and selecting
    - During anticipation work, considering how possible student responses are mathematically related

5. Connecting student responses during the discussion

- **It involves:**
    - Encouraging students to make mathematical connections between different student responses
    - Making the key mathematical ideas that are the focus of the lesson salient
- **It is supported by:**
    - Anticipating, monitoring, selecting, and sequencing
    - During planning, considering how students might be prompted to recognize mathematical relationships between responses

Why These Five Practices Likely to Help

Provides teachers with more control

- Over the content that is discussed
- Over teaching moves: not everything improvisation

Provides teachers with more time

- To diagnose students' thinking
- To plan questions and other instructional moves

Provides a reliable process for teachers to gradually improve their lessons over time

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Orchestrating Productive Mathematical Discussions: Five Practices for Helping Teachers Move Beyond Show and Tell

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ARTICLES

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Teachers who attempt to use inquiry-based, student-centered instructional tasks face challenges that go beyond identifying well-designed tasks and setting them up appropriately in the classroom. Because solution paths are usually not specified for these kinds of tasks, students tend to approach them in unique and sometimes unanticipated ways. Teachers must not only strive to understand how students are making sense of the task but also begin to align students' disparate ideas and approaches with canonical understandings about the nature of mathematics. Research suggests that this is difficult for most teachers (Ball, 1993, 2001; Leinhardt & Steele, 2005; Schoenfeld, 1998; Sherin, 2002). In this article, we present a pedagogical model that specifies five key practices teachers can learn to use student responses to such tasks more effectively in discussions: anticipating, monitoring, selecting, sequencing, and making connections between student responses. We first define each practice, showing how a typical discussion based on a cognitively challenging task could be improved through their use. We then explain how the five practices embody current theory about how to support students' productive disciplinary engagement. Finally, we close by discussing how these practices can make discussion-based pedagogy manageable for more teachers.

A key challenge that mathematics teachers face in enacting current reforms is to orchestrate whole-class discussions that use students' responses to instructional tasks in ways that advance the mathematical learning of the whole class (e.g., Ball, 1993; Lampert, 2001). In particular, teachers are often faced with a wide array of student responses to cognitively demanding tasks and must find ways to use them to guide the class toward deeper understandings of significant mathematics. Here, we propose a model for the effective use of student responses to such tasks in whole-class discussions that we argue has the potential for making such teaching manageable for many more teachers. Our model provides teacher educators with a set of five practices that they can use in their work with K–12 pre- and in-service teachers to help them learn how to orchestrate discussions that both build on student thinking and also advance important mathematical ideas. Researchers of classroom processes, teaching, and student learning of mathematics will also be interested in the five practices model as a way of conceptualizing investigations of classroom discourse.

We begin by discussing the importance and challenges of facilitating mathematical discussions that are launched through cognitively demanding mathematical tasks—problems that promote conceptual understanding and the development of thinking, reasoning, and problem-solving skills (Doyle, 1983, 1988; Henningsen & Stein, 1997; Hiebert & Wearne, 1993; Stein, Grover, & Henningsen, 1996). We then describe our five practices model using a concrete example of a classroom discussion that was not as effective as it could have been to illustrate how these practices can be used to more effectively facilitate mathematical discussions. Next, we ground the five practices in a theoretical framework for promoting productive disciplinary engagement to explain how the practices work together to help teachers create discussions that simultaneously build on student thinking while leading toward the development of important mathematical ideas. We close by discussing how the five practices model makes discussion facilitation more manageable for teachers.

THE IMPORTANCE AND CHALLENGES OF FACILITATING MATHEMATICAL DISCUSSIONS

Mathematical discussions are a key part of current visions of effective mathematics teaching (e.g., Cobb, Boufi, et al., 1997; Kazemi & Stipek, 2001; Nathan & Knuth, 2003). In several countries, including the United States, the expected role of the teacher is changing from “dispenser of knowledge” and arbiter of mathematical “correctness” to an engineer of learning environments in which students actively grapple with mathematical problems and construct their own understandings (Freudenthal, 1991; Gravemeijer, 1994; Lewis & Tsuchida, 1998; National Council of Teachers of Mathematics [NCTM], 1989, 1991; Stigler & Hiebert, 1999). In this vision, students are presented with more realistic and complex mathematical problems, use each other as resources for working through those problems, and then share their strategies and solutions in whole-class discussions that are orchestrated by the teacher. The role of the teacher during whole-class discussions is to develop and then build on the personal and collective sensemaking of students rather than to simply sanction particular approaches as being correct or demonstrate procedures for solving predictable tasks (e.g., Fennema et al., 1996). Such discussions are thought to support student learning of mathematics in part by helping students learn mathematical discourse practices (e.g., Chapin, O’Connor, & Anderson, 2003; Michaels et al., 2002), making students’ thinking public so it can be guided in mathematically sound directions (e.g., Forman et al., 1998), and encouraging students to construct and evaluate their own and each others’ mathematical ideas (e.g., Forman, McCormick, & Donato, 1998).

A typical reform-oriented lesson that incorporates these kinds of whole-class discussions often proceeds in three phases (Baxter & Williams, in press; Lampert, 2001; Sherin, 2002). It begins with the launching of a mathematical problem by the teacher that embodies important mathematical ideas and can be solved in multiple ways (e.g., Lampert, 2001; Stroup, Ares, & Hurford, 2005). During this “launch phase,” the teacher introduces the students to the problem, the tools that are available for working on it, and the nature of the products they will be expected to produce. This is followed by the “explore phase” in which students work on the problem, often discussing it in pairs or small groups. As students work on the problem, they are encouraged to solve the problem in whatever way makes sense to them and be prepared to explain their approach to others in the class. The lesson then concludes with a whole-class discussion and summary of various student-generated approaches to solving the problem. During this "discuss and summarize" phase, a variety of approaches to the problem are displayed for the whole class to view and discuss.

During what we call "the first generation" of practice and research that instantiated this vision, the role of the teacher with respect to building mathematical ideas was ill-defined. Emphasis was placed on the use of cognitively demanding tasks (e.g., Henningsen & Stein, 1997), the encouragement of productive interactions during the explore phase (e.g., Yackel et al., 1990), and the importance of listening respectfully to students' reasoning throughout (e.g., Fennema, Carpenter, & Peterson, 1989). During whole-class discussions, the focus tended to be on creating norms that would allow students to feel that their contributions were listened to and valued (e.g., Cobb, Wood, &Yackel, 1993) and on the kinds of teacher questions that would prompt students to explain their thinking (e.g., Hiebert & Wearne, 1993). Less attention was paid to what teachers could actively do to guide whole-class discussions toward important and worthwhile mathematics. In fact, many teachers got the impression that in order for discussions to be focused on student thinking, they must avoid providing any substantive guidance at all (cf. Baxter & Williams, in press; Chazen & Ball, 2001; Lobato, Clarke, & Ellis, 2005; Smith, 1996).

To provide an example of the kinds of discussions that often resulted during this first generation, and which still continue in many teachers' classrooms, consider the following vignette that characterizes the kinds of mathematical discussions that often occur in U.S. classrooms, even those using cognitively demanding tasks as their basis for whole-class discussions.[1]

Leaves and Caterpillars Vignette

Students in David Crane's fourth-grade class were solving the following problem: "A fourth-grade class needs five leaves each day to feed its 2 caterpillars. How many leaves would they need each day for 12 caterpillars?" Mr. Crane told his students that they could solve the problem any way they wanted, but emphasized that they needed to be able to explain how they got their answer and why it worked.

As students worked in pairs to solve the problem, Mr. Crane walked around the room making sure that students were on task and making progress on the problem. He was pleased to see that students were using lots of different approaches to the problem—making tables, drawing pictures, and, in some cases, writing explanations.

He noticed that two pairs of students had gotten wrong answers as shown below.

Darnell and Marcus	Missy and Kate
Answer: 60 5 Leaves X12 caterpillars 60 Leaves for caterpillars	Answer: 15 caterpillars They added 10 caterpillers, and so I added 10 leaves.

Mr. Crane wasn't too concerned about the incorrect responses, however, since he felt that once these students saw several correct solution strategies presented, they would see what they did wrong and have new strategies for solving similar problems in the future.

When most students were finished, Mr. Crane called the class together to discuss the problem. He began the discussion by asking for a volunteer to share their solution and strategy, being careful to avoid calling on the students with incorrect solutions. Over the course of the next 15 minutes, first Kyra, then Jason, Jamal, Melissa, Martin and Janine volunteered to present the solutions to the task that they and their partner had created. Their solutions are shown in Figure 1.

During each presentation, Mr. Crane made sure to ask each presenter questions that helped them to clarify and justify their work. He concluded the class by telling students that the problem could be solved in many different ways and now, when they solved a problem like this, they could pick the way they liked best because they all gave the same answer.

To some, this lesson would be considered exemplary. Indeed Mr. Crane did many things well, including allowing students to construct their own way of solving this cognitively challenging task[2] and stressing the importance of students' being able to explain their reasoning. However, a more critical eye might have noted that the string of presentations did not build toward important mathematical ideas. The upshot of the discussion appeared to be "the more ways of solving the problem the better," but, in fact, Mr. Crane only held each student accountable for knowing one way to solve the problem. In addition, although Mr. Crane observed students as they worked, he did not appear to use this time to assess what students understood about proportional reasoning (Nelson, 2001; Shifter, 2001) or to select particular students' work to feature in the whole-class discussion (Lampert, 2001; NCTM, 1991). And he gathered no information regarding whether the two pairs of students who had gotten the wrong answer (Darnell and Marcus, and Missy and Kate) were helped by the student presentations of correct strategies. Had they diagnosed the faulty reasoning underneath their approaches?

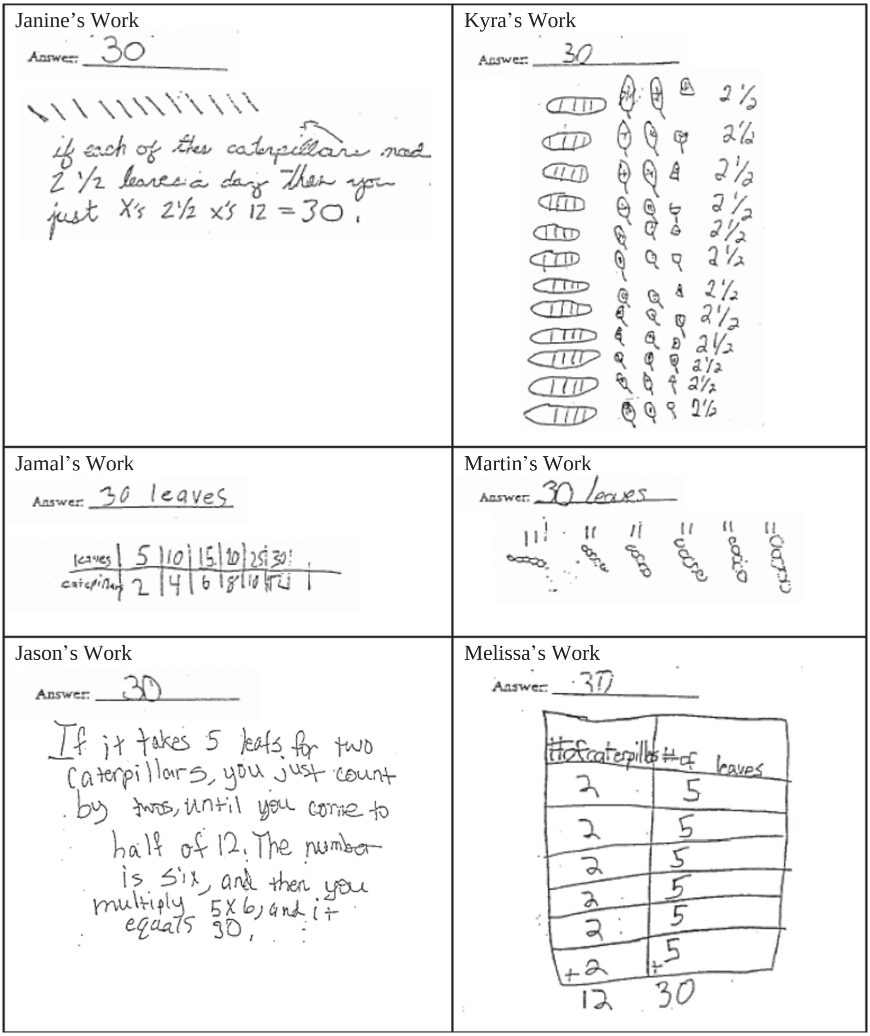

FIGURE 1 Student Work Presented in the Hypothetical Classroom of Mr. Crane during the Leaves and Caterpillar Vignette (Student Work from Smith, Hillen, & Heffernan, 2003).

In fact, we argue that much of the discussion in Mr. Crane's classroom was what Ball (2001) has called a "show and tell" in which students with correct answers each take turns sharing their solution strategies (see also, Wood & Turner-Vorbeck, 2001). There was little filtering by the teacher about which mathematical ideas each strategy helped to illustrate, nor any attempt to highlight those ideas (Lampert, 2001; Schoenfeld, 1998). In addition, the teacher did not draw connections among different solution methods or tie them to important disciplinary methods or mathematical ideas **(**Ball, 1993; Boaler & Humphries, 2005). Finally, there was no attention to weighing which strategies might be most useful, efficient, accurate, and so on, in particular circumstances (Nathan & Knuth, 2003). All were treated as equally good.

In short, providing students with cognitively demanding tasks with which to engage and then conducting "show and tell" discussions cannot be counted on to move an entire class forward mathematically. Indeed, this generation of practice was eventually criticized for creating classroom environments in which near-complete control of the mathematical agenda was relinquished to students (e.g., Ball, 1993, 2001; Chazen & Ball, 2001; Leinhardt, 2001). Some teachers went so far as to misperceive the appeal to honor students' thinking and reasoning as a call for a complete moratorium on teacher shaping of the quality of students' mathematical thinking. Due to the lack of guidance with respect to what teachers *could* do to encourage rigorous mathematical thinking and reasoning, many teachers were left feeling that they should avoid telling students anything (e.g., Baxter & Williams, in press; Chazen & Ball, 2001; Leinhardt, 2001; Lobato, Clarke, & Ellis, 2005; Smith, 1996; Windschitl, 2002).

A related criticism concerned the fragmented and often incoherent nature of the discuss-and-summarize phases of lessons. In these "show-and-tells," as exemplified in Mr. Crane's classroom, one student presentation would follow another with limited teacher (or student) commentary and no assistance with respect to drawing connections among the methods or tying them to widely shared disciplinary methods and concepts. There was no mathematical or other reason for students to necessarily listen to and try to understand the methods of their classmates. As illustrated in Mr. Crane's comment at the end of the class, students could simply "pick the way they liked best." This led to an increasingly recognized dilemma associated with inquiry- and discovery-based approaches to teaching: the challenge of aligning students' developing ideas and methods with the disciplinary ideas that they ultimately are accountable for knowing (e.g., Brown & Campione, 1994).

In a widely cited article, "Keeping an Eye on the Mathematical Horizon," Ball (1993) argued that the field needed to take responsibility for helping teachers to learn how to continually "size up" whether important mathematical ideas were being developed in these discussions and be ready to step in and redirect the conversation when needed. Unfortunately, guidance for how to do this remains scant. In this article, we join Ball (1993, 2001) and others (e.g., Gravemeijer, 2004; Lampert, 2001; Nathan & Knuth, 2003; Nelson, 2001; Wood & Turner-Vorbeck, 2001) who are seeking to identify ways in which teachers can effectively guide whole-class discussions of student-generated work toward important and worthwhile disciplinary ideas. We call this "second generation" practice and view it as a form of instruction that re-asserts the critical role of the teacher in guiding mathematical discussions. The hallmark of second generation practice is its focus on using student-developed work as the launching point of whole-class discussions in which the teacher actively shapes the ideas that students produce to lead them toward more powerful, efficient, and accurate mathematical thinking.

The literature now includes several compelling illustrations of what expert facilitators commonly do and must know to facilitate mathematical discussions that are, in the words of Ball (1993), accountable both to the discipline and to students (Ball, 2001; Chazen & Ball, 2001; Lampert, 2001; Leinhardt & Steele, 2005; Sherin, 2002). However, research has also demonstrated the significant pedagogical demands that are involved in orchestrating discussions that build on student thinking in this manner (e.g., Ball, 2001; Brown & Campione, 1994; Chazen & Ball, 2001; Lampert, 2001; Leinhardt & Steele, 2005; Schoenfeld, 1998; Sherin, 2002).

First, compared with presenting a lecture or conducting a recitation lesson in which mathematical procedures are demonstrated, facilitating a discussion around a task that can be solved in numerous ways greatly reduces teachers' degree of control over what is likely to happen in a lesson (e.g., Chazen, 2000). This can be particularly daunting for teachers who are new to discussion-based pedagogy, reducing their sense of efficacy for supporting student learning (Smith, 1996). In addition, many models of expert practice in the literature feature extremely skilled discussion facilitators: teachers who—with apparent ease—make rapid online diagnoses of students' understandings, compare them with desired disciplinary understandings, and then fashion an appropriate response. For teachers new to discussions or to the particular curriculum in which they are hoping to use them, achieving this level of improvisation can feel unattainable (Borko & Livingston, 1989; Heaton, 2000; Schoenfeld, 1998; Sherin, 2002). Indeed, research has shown that successful improvisation requires an extensive network of content knowledge, pedagogical knowledge, and knowledge of students as learners that is interwoven, and which is often limited for many teachers (e.g., Borko & Livingston, 1989; Margolinas, Coulange, & Bessot, 2005; Sherin, 2002).

Thus, as with other areas of expertise (e.g., Bransford, Brown, & Cocking, 1999; White & Frederiksen, 1993), experts are incomplete guides for teachers who want to learn how to become discussion facilitators (hereafter referred to as novices). While experts can help teachers to see the power of discussions that simultaneously honor both student thinking and a mathematical agenda, they often portray effective discussion facilitation as dauntingly complex while not addressing the novice's desire for easy-to-implement "how-to's" for learning how to facilitate such discussions. Moreover, they do not address the fact that novices have different knowledge bases than experts that affect the practices they can implement effectively. Without the expert's reservoir of knowledge about how to relate student responses to important mathematical content, novices cannot improvise their way through such discussions as experts do (Schoenfeld, 1998). Without solid expectations for what is likely to happen, novices are regularly surprised by what students say and do, and therefore often do not know how to respond to students in the midst of a discussion. They feel out of control and unprepared, which then reduces their efficacy as teachers, making discussion-based pedagogy a lot less attractive (Smith, 1996).

Instead, we argue that novices need a set of practices they can do to both prepare them to facilitate discussions (Ghousseini, 2007; Lampert, 2007) and help them gradually and reliably learn how to become better discussion facilitators over time (Fernandez & Yoshida, 2004; Hiebert, Morris, & Glass, 2003; Stigler & Hiebert, 1999). We propose our model of the five practices as one such tool, which is designed specifically for whole-class discussions that are conducted after students work on high-level cognitively-challenging tasks (Stein et al., 2000).

FIVE PRACTICES FOR FACILITATING MATHEMATICAL DISCUSSIONS AROUND COGNITIVELY DEMANDING TASKS

In our model of five practices for discussion facilitation, the intent is to make discussion facilitation something that is manageable for novices, those teachers who are new to this form of teaching. We do this by purposely de-emphasizing the improvisational aspects of discussion facilitation in favor of a focus on those aspects of mathematical discussions that can be planned for in advance (cf. Fennema & Franke, 1992; Gravemeijer, 2004). Through planning, teachers can anticipate likely student contributions, prepare responses they might make to them, and make decisions about how to structure students' presentations to further their mathematical agenda for the lesson. By expanding the time to make an instructional decision from seconds to minutes (or even hours) our model allows increasing numbers of teachers to feel—and actually be—better prepared for discussions.

Specifically, the five practices are: (1) anticipating likely student responses to cognitively demanding mathematical tasks, (2) monitoring students' responses to the tasks during the explore phase, (3) selecting particular students to present their mathematical responses during the discuss-and-summarize phase, (4) purposefully sequencing the student responses that will be displayed, and (5) helping the class make mathematical connections between different students' responses and between students' responses and the key ideas. Each practice has been discussed separately by various authors; our contribution here is to integrate them into a single package. As shown in Figure 2, we view each of the practices as drawing on the fruits of the practices that came before it. For example, teachers'

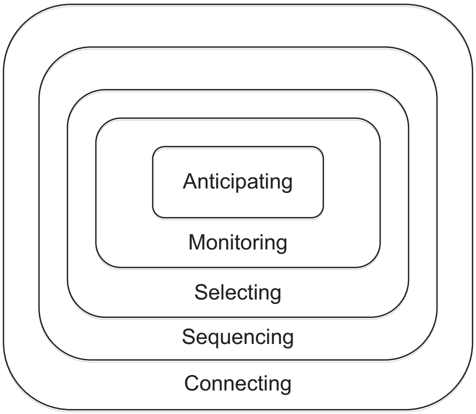

FIGURE 2 Schematic Diagrams of the Five Practices in which Each Practice Depends on the Practices Embedded within it.

monitoring of students work during the explore phase will benefit from their pre-class preparation of anticipating how students might approach the tasks. Similarly, the practice of selecting particular students to present their work will benefit from careful monitoring of the range of responses that students produce during the explore phase. In addition, successfully using the five practices depends on implementing a cognitively demanding instructional task with multiple possible responses and having well-defined instructional goals, both of which are supported by teachers' understanding of their students' current mathematical thinking and practices.

Together, we feel these practices help make it more likely that teachers will be able to use students' responses to advance the mathematical understanding of the class as a whole. Each practice is described in more detail below, with the example of Mr. Crane's discussion illustrating how each could have contributed to a more productive mathematical discussion in his class.

Anticipating Students' Mathematical Responses

The first practice is for teachers to make an effort to actively envision how students might mathematically approach the instructional tasks(s) that they will be asked to work on (e.g., Fernandez & Yoshida, 2004; Lampert, 2001; Schoenfeld, 1998; Smith, 1996; Stigler & Hiebert, 1999). This involves much more than simply evaluating whether a task will be at the right level of difficulty or of sufficient interest to students, and it goes beyond considering whether they are likely to get the "right answer." Anticipating students' responses involves developing considered expectations about how students might mathematically interpret a problem, the array of strategies—both correct and incorrect—they might use to tackle it, and how those strategies and interpretations might relate to the mathematical concepts, representations, procedures, and practices that the teacher would like his or her students to learn (Lampert, 2001; Schoenfeld, 1998; Yoshida, 1999, cited in Stigler & Hiebert, 1999).

Consider how using the practice of anticipating might have affected the nature of discussion in Mr. Crane's class. Table 1 shows the variety of strategies that students might use to solve the Caterpillar and Leaves problem (first column); the names of the students whose work illustrated those strategies, including the representation they used (second column); and how the strategies relate to one another (third column).

As shown in the first two columns of the first row, anticipation would have enabled Mr. Crane to recognize that the response given by Missy and Kate reflects a common misconception that students of this age have with respect to proportionality: they identify the relationship between the quantities, here the numbers of caterpillars and leaves, as additive rather than multiplicative (e.g., Cramer, Post, & Currier, 1993; Hart, 1988; Noelting, 1980). Anticipating this in advance would have made it possible for Mr. Crane to have a question ready to ask or an activity that the students could do that might have helped them and other students recognize why this approach, though tempting, does not make sense.

Anticipation requires that teachers, at a minimum, actually do the mathematical tasks that they are planning to ask their students to do. However, rather than finding a single strategy to solve a problem, teachers need to devise and work through as many different solution strategies as they can. Moreover, if they put themselves in the position of their students while doing the task, they can anticipate some of the strategies that students with different degrees of mathematical sophistication are likely to produce and consider ways that students might misinterpret problems or get confused along the way, as some of Mr. Crane's students did. Each time they use a task, teachers can add to their fund of knowledge about likely student responses.

In addition to drawing on their knowledge of their particular students' mathematical skills and understandings, teachers might draw on their knowledge of the research literature about typical student responses to the same or similar tasks or of common student understandings of related concepts and procedures (e.g., Fennema et al., 1996) . The practice of anticipating student responses can be further supported when teachers use mathematics curricula that include typical student responses to problems, as is done in many Japanese curricula (Fernandez & Yoshida, 2004; Schoenfeld, 1998; Stigler & Hiebert, 1999) and in some American curricula (e.g., Russell, Tierney, Mokros, & Economopoulos' 2004, *Investigations in Number, Data, and Space*). In addition, there is a growing library of written and video cases of mathematics teaching designed for teachers that often

TABLE 1
Strategies that Students Might Use, Students Whose Work Illustrated the Strategies, and Connections Between the Strategies

Strategies Students Might Use	*Student/ Representation*	*Connections between Strategies*
1. Additive (incorrect) Find that the difference between the original number of caterpillars (2) and the new number of caterpillars (12). The difference is 10. Then add 10 to the original number of leaves (5) to find the new number of leaves (15).	Missy & Kate/written explanation	All three solutions (Missy & Kate, Martin and Melissa) use addition but Missy and Kate do not keep the ratio between caterpillars and leaves (2 to 5) constant while both Martin and Melissa do.
2. Scaling up–replicating sets Keep replicating sets of 2 caterpillars and 5 leaves until you have 12 caterpillars. Then count the number of leaves and it will be 30.	Martin/picture Melissa/table	Martin uses pictures to show his replication of sets while Melissa uses a table.
3. Scaling up–growing the ratio "Grow" the number of leaves and the number of caterpillars by continuing to increase the number of caterpillars by 2 and the number of leaves by 5 through addition or multiplication. Once you reach 12 caterpillars through this process you will find that you have 30 leaves.	Jamal/table (addition or multiplication)	Jamal, like Melissa, uses a table. However, his table shows the number of caterpillars and leaves "growing" by the same amount. Rather than having exactly the same entry in each row (like Melissa's) in Jamal's table each column has 2 caterpillars and 5 leaves more than the previous column.

4. Unit rate Determine that 1 caterpillar eats 2.5 leaves each day. Then find the number of leaves eaten by 12 caterpillars by multiplying by 2.5. This will give you 30 leaves.	Kyra/picture (repeated addition) Janine/picture and written explanation (multiplication)	Both Kyra and Janine found the amount of leaves eaten by one caterpillar per day. It appears that Kyra used addition while Janine used multiplication. The ratio of 1 caterpillar to 2.5 leaves can be compared to the ratio of 2 caterpillars to 5 leaves used by Melissa, Martin, and Jamal. 2.5 can be seen in Jamal’s table as the factor that relates the numbers in the caterpillar row to the numbers in the leaves row.
5. Scale factor Determine that the new number of caterpillars (12) is 6 times the original number of caterpillars (2). Then find the new number of leaves by multiplying the original number of leaves (5) by six to get 30 leaves.	Jason/numeric-narrative (multiplication)	Jason has found that the scale factor that relates the original amounts to the new amounts. This scale factor is the same as the number of sets constructed by Martin and Melissa. It is also the same as the number of entries in Jamal’s table.

include extensive information about student responses to the instructional tasks that are the basis of the lessons in the cases (e.g., Barnett, Goldstein, & Jackson, 1994; Boaler & Humphries, 2005; Smith, Silver, Stein, Boston, & Henningsen, 2005; Smith, Silver, Stein, Boston, Henningsen, & Hillen, 2005; Smith, Silver, Stein, Henningsen, Boston, & Hughes, 2005; Stein et al., 2000). Building on such preexisting resources provides especially helpful scaffolding for teachers who are new to conducting whole-class discussions around cognitively demanding mathematical tasks.

Monitoring Student Responses

Monitoring student responses involves paying close attention to the mathematical thinking in which students engage as they work on a problem during the explore phase (e.g., Brendehur & Frykholm, 2000; Lampert, 2001; Nelson, 2001; Schoenfeld, 1998; Shifter, 2001). This is commonly done by circulating around the classroom while students work (e.g., Baxter & Williams, in press; Boerst & Sleep, 2007; Lampert, 2001). The goal of monitoring is to identify the mathematical learning potential of particular strategies or representations used by the students, thereby honing in on which student responses would be important to share with the class as a whole during the discussion phase (Brendehur & Frykholm, 2000; Lampert, 2001). For example, rather than only noting how many students are actually working on the problem or who seems to be frustrated, teachers should also attend to the mathematical ideas that are in play in their work and talk. That is, teachers should actively attend to the mathematics within what students are saying and doing, assess the mathematical validity of students' ideas, and make sense of students' mathematical thinking even when something is amiss (Nelson, 2001; Shifter, 2001).

Those teachers who have made a good faith effort during initial planning to anticipate how students might respond to a problem will feel better prepared to monitor what students actually do during the explore phase (Lampert, 2001; Schoenfeld, 1998). Still, this can be challenging, especially if the strategies or representations used by students are unfamiliar to the teacher (Ball, 2001; Crespo, 2000; Shifter, 2001; Wallach & Even, 2005). One way to manage the challenge is for teachers to jot down notes about the particular approaches and reasoning strategies that students are using. In addition, some tasks involve manipulatives, representations, response sheets, or computer-based records that make it possible to identify students' strategies by visually examining what they have done with these materials. In other cases, teachers can assess students' mathematical thinking by listening to a group's conversations as they work, making sure to hear "below the surface" features of students' talk and representations so as to see the mathematical promise in what students are doing and thinking, and by asking students probing questions. It is also important for teachers to ask questions that will help them assess students' mathematical thinking—in particular students' understanding of key concepts that relate to the goal of the lesson. Such monitoring is further supported when students have been taught representational or communicative practices that will make their mathematical thinking more accessible to others (e.g., see Lampert, 2001).

Returning to the Leaves and Caterpillars Vignette, we note that while Mr. Crane did circulate around the classroom and understood both who had and had not gotten correct answers and that a range of representations (tables, pictures, etc.) had been used, the lack of organization of his sharing at the end of the class suggests he had not monitored the specific mathematical learning potential available in any of the students' responses. For example, Mr. Crane did not recognize (or at least make use of) the fact that both Kyra's and Janine's solutions were based on the concept of unit rate (see row 4 of Table 1) as they figured out that each caterpillar eats 2.5 leaves per day, while Jason's solution made use of the concept of a scale factor (see row 5 of Table 1) when he reasoned that because the number of caterpillars scaled up by a factor of 6, the number of leaves also would have to scale up by this same multiplicative factor (Cramer & Post, 1993; Lesh, Post, & Behr, 1988). Knowing this would have allowed Mr. Crane to have the class generalize from these students' approaches to introduce these key mathematical ideas about proportionality to the class.

In general, by taking time during the explore phase to monitor the mathematical basis behind students' responses, Mr. Crane would have had more information on which to guide his instructional decisions during the whole class discussion and beyond. In addition, working to understand students' solutions as much as possible during the explore phase would have give him minutes rather than simply seconds to decide how to respond to students' mathematical ideas during the discussion. Finally, as we will now discuss, Mr. Crane and other teachers can use the information that they obtain about student thinking during monitoring to plan which responses they will feature in the ensuing class discussion. As Lampert (2001, p. 140, emphasis ours) summarized it, "If I watch and listen during small-group independent work, I am then able to use my observations to decide *what* and *who* to make focal" during whole-class discussion.

Purposefully Selecting Student Responses for Public Display

Having monitored the available student responses in the class, the teacher can then select particular students to share their work with the rest of the class in order to get "particular piece[s] of mathematics on the table" (Lampert, 2001, p. 146; also see Stigler & Hiebert, 1999). A typical way to do this is to call on specific students (or groups of students) to present their work as the discussion proceeds. Alternatively, a teacher might ask for volunteers but then select a particular student that he or she knows is one of several who has a particularly useful idea to share with the class. This is one way of balancing the tension between "keeping the discussion on track and allowing students to make spontaneous contributions that they consider … to be relevant" (Lampert, 2001, p. 174). Still, in all these methods of selecting, the teacher remains in control of which students present their strategies, and therefore what the mathematical content of the discussion will likely be.

Returning to the Leaves and Caterpillar Vignette, if we look at the strategies that were shared, we noted earlier that both Kyra and Janine had similar strategies that used the idea of a unit rate (see row 4, Table 1). Given that, there may not have been any added value for students' mathematical learning in having both be shared. In fact, if Mr. Crane wanted the students to see the multiplicative nature of the relationship between the leaves and caterpillars (Vergnaud, 1988), he might have selected Jason (see row 5, Table 1) to present his strategy as it most clearly involved multiplication.

Rather than placing the class and the teacher at the mercy of whatever strategies student volunteers might present, the purposeful selection of presenters makes it more likely that important mathematical ideas will be discussed by the class. The teacher selects students to present whose strategies depend on those ideas, allowing the ideas to be illustrated, highlighted, and then generalized. Teachers can also ensure that common misconceptions are aired publicly, are understood, and are corrected by selecting students like Missy and Kate to present strategies and relying on them so that the class as a whole can then examine them in order to understand why and how the reasoning does not work (Confrey, 1990). And, if necessary, a teacher can introduce a particularly important strategy that no one in the class has used by sharing the work of students from other classes (e.g., Boaler & Humphreys, 2005; Schoenfeld, 1998) or offering one of his or her own for the class to consider (e.g., Baxter & Williams, in press). Another way that teachers can increase the repertoire of strategies available for public sharing is to offer instructional support during the explore phase to students who appear to be on the verge of implementing a unique and important approach to solving the problem, but who need some help to be able to actually achieve that and effectively share it with their classmates.

At the same time that care is taken to have certain responses publicly aired, other responses might be avoided altogether or presented at a later point in time when the class can more effectively deal with them (Schoenfeld, 1998). Similarly, a response that is important but unexpected by the teacher can be delayed from consideration until a later class when the teacher has had more time to think about the mathematics underlying that response (Engle, 2004). Such revisiting of students' work and the ideas behind it is a particularly effective tool for newer teachers who can then consult with their colleagues, teacher education instructors, curriculum materials, and other resources for deepening their understanding of the mathematics that they are teaching and how students tend to think about it.

It is important, however, that teachers do not simply use the technique of selecting to avoid dealing with those students or mathematical ideas that they have more difficulty teaching. One way to avoid this is for teachers to regularly review their monitoring notes to identify any patterns in who was called on and who was not, and which ideas were discussed and which were not. Teachers can then adjust their future practices accordingly, preparing to make and take up opportunities to select those students and ideas that have not gotten as much consideration as they should have.

Purposefully Sequencing Student Responses

Having selected particular students to present, the teacher can then make decisions about how to sequence the students' presentations with respect to each other (Schoenfeld, 1998; West, 1994). By making purposeful choices about the order in which students' work is shared, teachers can maximize the chances that their mathematical goals for the discussion will be achieved. For example, the teacher might want to have the strategy used by the majority of students presented before those that only a few students used to help validate the work that students did and make the beginning of the discussion accessible to as many students as possible (West, 1994). This can allow students to build a depth of understanding of the problem that will be helpful later for making sense of more unique or complex solution strategies. Similar benefits can be had by starting a discussion with a particularly easy-to-understand strategy like Mr. Crane could have done with Martin's picture, which concretely depicted how the number of leaves and caterpillars increased proportionally.

Another possibility for sequencing is to begin with a common strategy that is based on a misconception that several students had so the class can clear up that misunderstanding in order to be able to work on developing more successful ways of tackling the problem (for an example of this in action see "The Case of Marie Hanson" in Smith, Silver, Stein, Boston, Henningsen, & Hillen, 2005). For example, there might have been some payoff for Mr. Crane in first sharing the solution produced by Missy and Kate and then following it with the solution produced by Melissa, who also used addition but did so in a way that preserved proportionality between the quantities.

In addition, the teacher might want to have related or contrasting strategies be presented right after one another to make it easier for the class to mathematically compare them. For example, in his mathematical problem-solving course, Schoenfeld (1998, p. 68) sometimes had students discuss particular problem solving approaches not "in the order that they had been generated, but in an order that allow[s] various mathematical 'lessons' to emerge more naturally from the discussions." In the case of Mr. Crane's discussion, here is one reasonable sequencing that he might have considered using:

1. Martin: picture (scaling up–replicating sets)
2. Jamal: table (scaling up–growing the ratio)
3. Janine: picture and written explanation (unit rate)
4. Jason: written explanation (scale factor)

This sequencing begins with two different but relatively easy-to-understand scaling up strategies and ends with a fairly sophisticated scale factor strategy, which would likely support the goal of accessibility. In addition, by having the same relatively accessible strategy—scaling up—be presented with two different representations, this could support the goal of helping students to better understand this particular strategy and the relationship between these representations, in this case a model of the problem situation in Martin's picture and Jamal's somewhat more abstract tabular representation.

Thus, rather than being at the mercy of when students happen to contribute an idea to a discussion, teachers can select students to present in a particular sequence to make a discussion more mathematically coherent and predictable. However, as our list of possible sequences that Mr. Crane might have used indicated, much more research needs to be done to compare these and other possible sequencing methods with each other in order to understand what each best contributes. However, it is clear that as with the other four practices, what particular sequence teachers choose to use should depend crucially on both teachers' knowledge of their students and their particular instructional goals.

Connecting Student Responses

Finally, teachers can help students draw connections between the mathematical ideas that are reflected in the strategies and representations that they use (e.g., Ball, 2001; Boaler & Humphreys, 2005; Brendehur & Frykholm, 2000). They can help students make judgments about the consequences of different approaches for the range of problems that can be solved, one's likely accuracy and efficiency in solving them, and the kinds of mathematical patterns that can be most easily discerned. They also can help students see how the same powerful idea (e.g., there is a multiplicative relationship between quantities in a ratio) can be embedded in two strategies that on first glance look quite dissimilar (e.g., one performed using a picture/written explanation, another with a table; as in Janine's and Jamal's work in Figure 2). So, rather than having mathematical discussions consist of separate presentations of different ways to solve a particular problem, the goal is to have student presentations build on each other to develop powerful mathematical ideas.

Returning to Mr. Crane's class, after having students compare and contrast the use of scaling up in Martin's picture and Jamal's table, Mr. Crane might consider comparing Jamal and Janine's responses. As shown in Table 1, Janine's unit rate of 2.5 can be discerned in Jamal's table by dividing the entry in the number of leaves column by the entry in the number of caterpillars column. This would help students to generalize the concept of a unit rate as something that can be seen across multiple mathematical representations. Similarly, Mr. Crane's students could be asked to compare Jason's work with both Jamal's and Martin's to see that Jason's scale factor of 6 is the same as the number of sets constructed by Martin and the number of entries in Jamal's table. Using Jamal's table as a basis, it could become increasingly clear to students the ways in which unit rates and scale factors differ from but still relate to each other in a proportional situation. In general, if Mr. Crane's instructional goal for this lesson was having students flexibly understand different approaches—scale factor, scaling up, and unit rate—then having the students identify each of these ideas in each representation would be an especially worthwhile kind of connection that could be made.

However, there are many different ways that teachers might help a class draw connections besides what we have specifically suggested for Mr. Crane's lesson. In the transition between two students' presentations, a teacher can allude to some of the ways that the two students' strategies and mathematical ideas might be similar to or different from each other in the types of representations, operations, and concepts that were used (Hodge & Cobb, 2003). Or teachers can ask students to identify what is similar or different in the two presentations. All of these ways of helping students to connect their mathematical responses with each other can help make discussions more coherent. At the same time, doing this can prompt students to reflect on other students' ideas while evaluating and revising their own (Brendehur & Frykholm, 2000; Engle & Conant, 2002).

Finally, teachers could plan additional lessons in which the demands of the task might increase. For example, they may want to alter the initial problem to discuss issues of efficiency and how different strategies may be best suited for different problems. For example, students could be asked to determine how many leaves would be needed each day for 50 or 100 caterpillars. While all of the strategies used to correctly solve the initial problem may be considered reasonable (given the relatively small numbers involved), strategies such as ones used by Kyra, Martin, and Melissa become much less efficient as the numbers increase. Similarly, the teacher may want to alter the problem so the unit rate is more difficult to use (e.g., If 5 caterpillars eat 13 leaves then how many leaves will 100 caterpillars eat?). In this case, the approach used by Janine and Kyra might be more difficult since each caterpillar is now eating 26/10 leaves per day—a much more difficult fraction part to work with, while the approach used by Jason, finding the scale factor, would be much easier since it is an integer. Ultimately, the teacher may want to ask questions that require more flexible use of knowledge such as, "How many caterpillars could eat for a day on 100 leaves?"

GROUNDING OF THE FIVE PRACTICES IN A THEORY FOR PROMOTING PRODUCTIVE DISCIPLINARY ENGAGEMENT

In this section, we step back from the example of Mr. Crane's class to further explore the conceptual grounding of the five practices. We do so by situating them in a theoretical frame that addresses how productive disciplinary engagement can be supported in the classroom (Engle & Conant, 2002).

Since the advent of more student-centered, inquiry-based forms of instructional practice, teachers have struggled with how to orchestrate discussions in ways that both engage students' sense-making in authentic ways and move the class as a whole toward the development of important and worthwhile ideas in the discipline. Two norms that teachers can embody in their classrooms to address this challenge are student *authority* and *accountability* to the discipline (Engle & Conant, 2002). The idea behind student authority is that learning environments should be designed so that students are "authorized" to solve mathematical problems for themselves, are publicly credited as the "authors" of their ideas, and develop into local "authorities" in the discipline (see also Hamm & Perry, 2002; Lampert, 1990b; Scardamalia, Bereiter, & Lamon, 1994; Wertsch & Toma, 1995). A learning environment embodying the norm of accountability to the discipline regularly encourages students to account for how their ideas make contact with those of other mathematical authorities, both inside and outside the classroom (see also Boero et al., 1998; Cobb, Gravemeijer et al., 1997; Lampert, 1990a; Michaels et al., 2002).

At the heart of the challenge associated with student-centered practice is the need to strike an appropriate balance between giving students authority over their mathematical work and ensuring that this work is held accountable to the discipline. Nurturing students' mathematical authority depends on the opportunity for students to publicly engage in the solving of real mathematical problems, actively grapple with various strategies and representations that they devise to make headway on the problems, and judge the validity and efficacy of their own approaches themselves (Engle & Conant, 2002; Hiebert et al., 1996). A launch-explore-discuss lesson structure that uses cognitively demanding tasks with more than one valid mathematical solution strategy tends to be very effective at supporting students' authority. Individually and in small groups students have opportunities to solve problems in their own ways and then they become recognized as the authors of those approaches as they share them in small groups and with the class as a whole.

However, at the same time, the teacher must move students collectively toward the development of a set of ideas and processes that are accountable to the discipline—those that are widely accepted as worthwhile and important in mathematics as well as necessary for students' future learning of mathematics in school. Otherwise, the balance tips too far toward student authority and classroom discussions make insufficient contact with disciplinary understandings. Unfortunately, there is nothing in the launch-explore-discuss lesson structure in and of itself that particularly engenders such accountability as examples of “show and tell” style discussions like the Leaves and Caterpillar Vignette demonstrate.

On the other hand, efforts to encourage students to be accountable to the discipline can easily lead teachers to unwittingly undermine students’ authority and engaged sense-making (e.g., Elmore, Peterson, & McCarthy, 2000; Engle & Faux, 2006; Hamm & Perry, 2002). This may have happened in Mr. Crane’s class if he continued to limit student presentations just to those that reached the correct answer. This would have sent the message to students that strategies need to be validated by the teacher deciding to select them for presentation rather than through a process of mathematical reasoning in which students can participate. Similarly, teachers who do allow the discussion of incorrect strategies may still undermine students’ authority to evaluate the sensibleness of their and others’ ideas when they give subtle cues to their evaluations through differences in pauses, facial expressions, elaboration, and questioning of student responses of different levels of quality. In such cases, students tend to no longer report what they actually think about a problem, but instead what they believe their teachers will respond favorably to. Other teachers, concerned when their students reveal misunderstandings, often find it difficult to resist the temptation to directly correct students’ answers, which can further undermine students’ mathematical authority for using their own mathematical reasoning to evaluate the sensibleness of their own and others’ ideas.

The model described herein supports accountability to the discipline without undermining students’ mathematical authority through a set of teaching practices that take students’ ideas as the launching point, but shape class discussions so that over time important mathematical ideas are surfaced, contradictions exposed, and understandings developed or consolidated. Building on the resources provided by a variety of student responses to cognitively demanding tasks, the teacher selects particular responses to be discussed in a particular order that will support his or her instructional goals for students’ mathematical development.

While explicit and tractable to the teacher, these practices and their impact on the shape of discussions are largely invisible to students. Students do not see teachers doing their anticipating work in advance of the lesson, they do not know exactly what they are doing while circulating around the room, and they may not be fully aware of the basis behind teachers’ decisions about which strategies to have presented and in which order. Thus, these teacher actions do not untowardly impinge on students’ own growing mathematical authority in the ways that teacher hints and corrections do. It is students’ ideas that provide the fodder for discussions, with students publicly serving as the primary evaluators of them. At the same time, careful selection and steering has been done by the teacher—mostly under the radar—to move the class discussion in particular, mathematically productive directions. Thus, students can experience the magic of learning through interaction and communication with their peers, the exhilaration of co-constructing something new, and the payoff that comes from sustained listening and thinking in a concentrated and focused manner. At the same time, the discipline of mathematics has been represented through the teacher's wise selection of student ideas to discuss in a particular order and by prompts for students to make important mathematical connections between them.

MAKING MATHEMATICAL DISCUSSIONS MORE MANAGEABLE FOR TEACHERS

The premise underlying this article is that the identification and use of the five practices can make student-centered approaches to mathematics instruction more accessible to and manageable for more teachers. By giving teachers a roadmap of things that they can do in advance and during whole-class discussions, these practices have the potential for helping teachers to more effectively orchestrate discussions that are responsive to both students and the discipline. While the discussion herein featured an example from a fourth-grade classroom, the model we are proposing can be used by teachers at all levels K–12 and by teacher educators who are engaging their students in the discussion of a cognitively challenging mathematical task.

In addition, we argue that the five practices can also help teachers gain a sense of efficacy over their instruction (Smith, 1996) as they learn that there are ways for them to reliably shape students' discussions. In addition, teachers can be confident that each time that they use the five practices with a particular task, the discussion based on that task is likely to get more mathematically sophisticated. In fact, we have encouraged teachers to think about the five practices as a method for slowly improving the quality of discussions over time as their reservoir of experiences with specific tasks grows. For example, the first time a teacher uses a particular instructional task, he or she may focus on anticipating and monitoring in order to learn more about how his or her students tend to respond to the task and what mathematical ideas can be brought forth from students' responses. The second time around the teacher can use that information to make judicious choices about which approaches to be sure to select for class discussion. In later lessons, the teacher can use the information gathered in the previous go-arounds to begin developing effective methods of sequencing and connecting. Thus, over time a teacher's facilitation of a discussion around a particular task can improve, with the speed of progress accelerating if he or she works with other teachers, makes use of resources from research and curriculum materials, and consistently builds on records of what he or she observed and learned during each effort.

The five practices should not be viewed as a "stand-alone" remedy for the improvement of mathematics instruction. Rather they are one—albeit an important—component of effective pedagogical practice. As such, the practices need to be embedded in classroom norms that support inquiry learning, including respect for others' efforts and valuing the processes involved in mathematical argumentation.

The practices are also not a comprehensive prescription for mathematics learning. Learning mathematics well results from engagement in a sequence of carefully planned and orchestrated lessons, in addition to polishing the pedagogy surrounding individual tasks. As Hiebert and colleagues (1997, p. 31) have argued, "Teachers need to select sequences of tasks, so that, over time, students' experiences add up to something important." While our focus here is on preparing for and carrying out a single discussion, we recognize that such discussions must be viewed as part of a larger, coherent, and comprehensive curriculum.

Thus, the five practices do not provide an instant fix for mathematics instruction. Instead, they provide something much more important: a reliable process that teachers can depend on to gradually improve their classroom discussions over time. Along with others, we are coming to believe that the most practical visions for deeply and pervasively reforming mathematics teaching are those that support such slow and steady progress (Fernandez & Yoshida, 2004; Hiebert et al., 2003; Stigler & Hiebert, 1999), and we offer the five practices here as one helpful tool for realizing that vision for classroom discussions.

NOTES

1. This vignette is a composite of one type of discussion that we have regularly observed in mathematics classrooms using cognitively demanding tasks. Although constructed around actual student work (Smith, Hillen, & Heffernan, 2003), the specific events are hypothetical. However, the purpose of the vignette is not to serve as data but instead to illustrate our ideas while illuminating their practical import.
2. The Leaves and Caterpillars task is cognitively demanding for fourth graders as evidenced by both assessment results and an analysis of the task according to the Mathematical Tasks Framework (Stein, Smith, Henningsen, & Silver, 2000). In the seventh administration of National Assessment of Educational Progress (NAEP), only 6% of fourth graders gave a correct answer with a correct explanation to this task, while another 7% either gave a correct answer within an explanation or showed a correct method with a computational error (Kenney & Linquist, 2000). Most of the students (86%) gave an incorrect response. From the perspective of the Mathematical Tasks Framework, this task was implemented as a higher-level "Doing Mathematics" task most notably because "there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example," the task "requires students to access relevant knowledge and experiences and make appropriate use of them," and Mr. Crane required students to mathematically justify their solution methods (Stein, Smith, Henningsen, & Silver, 2000, p. 16; see also Engle & Adiredja, 2008).

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WARM UP PROBLEM

A fourth-grade class needs five leaves each day to feed its 2 caterpillars. How many leaves would the students need each day for 12 caterpillars?

Use drawings, words, or numbers to show how you got your answer.

- **Please try to do this problem in as many ways as you can, both correct and incorrect**

- **If done, share your work with a neighbor or look at the student work back side of the handout.**

Elementary Mathematics Specialists and Teacher Leaders Project
November 18, 2011

5 Practices for Orchestrating Productive Mathematical Discussions

Peg Smith
University of Pittsburgh

The Importance of Discussion

Mathematical discussions are a key part of current visions of effective mathematics teaching

- To encourage student construction of mathematical ideas
- To make student's thinking public so it can be guided in mathematically sound directions
- To learn mathematical discourse practices

Overview

- Analyze an instructional episode that will provide a basis for talking about classroom discussions
- Describe 5+ practices that you can learn in order to facilitate discussions more effectively and ground each in the instructional episode
- Discuss how the 5 practices could help improve teaching

Overview

- Analyze an instructional episode that will provide a basis for talking about classroom discussions
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Leaves and Caterpillar Vignette

- What aspects of Mr. Crane's instruction would you want him to see as promising (*reinforce*)?

- What aspects of Mr. Crane's instruction would you want to help him to work on (i.e., *refine*)?

Leaves and Caterpillar Vignette
What is Promising

- Students are working on a mathematical task that appears to be both appropriate and worthwhile
- Students are encouraged to provide explanations and use strategies that make sense to them
- Students are working with partners and publicly sharing their solutions and strategies with peers
- Students' ideas appear to be respected

Leaves and Caterpillar Vignette *What Can Be Improved*

- Beyond having students use different strategies, Mr. Crane's goal for the lesson is not clear
- Mr. Crane observes students as they work, but does not use this time to ***assess what students seem to understand*** or ***identify which aspects of students' work to feature*** in the discussion in order to make a mathematical point
- There is a "show and tell" feel to the presentations
  - not clear what each strategy adds to the discussion
  - different strategies are not related
  - key mathematical ideas are not discussed
  - no evaluation of strategies for accuracy, efficiency, etc.

Some Sources of the Challenge in Facilitating Discussions

- Lack of familiarity
- Reduces teachers' perceived level of control
- Requires complex, split-second decisions
- Requires flexible, deep, and interconnected knowledge of content, pedagogy, and students

Purpose of the Five Practices

To make student-centered instruction more manageable by moderating the degree of improvisation required by the teachers and during a discussion.

Overview

- Analyze an instructional episode that will provide a basis for talking about classroom discussions
- Describe 5+ practices that you can learn in order to facilitate discussions more effectively and ground each in the instructional episode
- Discuss how the 5 practices could help improve teaching

The Five Practices (+)

1. **Anticipating** (e.g., Fernandez & Yoshida, 2004; Schoenfeld, 1998)
2. **Monitoring** (e.g., Hodge & Cobb, 2003; Nelson, 2001; Shifter, 2001)
3. **Selecting** (e.g., Lampert, 2001; Stigler & Hiebert, 1999)
4. **Sequencing** (e.g., Schoenfeld, 1998)
5. **Connecting** (e.g., Ball, 2001; Brendehur & Frykholm, 2000)

The Five Practices (+)

0. **Setting Goals and Selecting Tasks**
1. **Anticipating** (e.g., Fernandez & Yoshida, 2004; Schoenfeld, 1998)
2. **Monitoring** (e.g., Hodge & Cobb, 2003; Nelson, 2001; Shifter, 2001)
3. **Selecting** (e.g., Lampert, 2001; Stigler & Hiebert, 1999)
4. **Sequencing** (e.g., Schoenfeld, 1998)
5. **Connecting** (e.g., Ball, 2001; Brendehur & Frykholm, 2000)

0$_1$. Setting Goals

- **It involves:**
  - Identifying what students are to know and understand about mathematics as a result of their engagement in a particular lesson
  - Being as specific as possible so as to establish a clear target for instruction that can guide the selection of instructional activities and the use of the five practices
- **It is supported by:**
  - Thinking about what students will come to know and understand rather than only on what they will do
  - Consulting resources that can help in unpacking big ideas in mathematics
  - Working in collaboration with other teachers

Mr. Crane’ s Class

Implied Goal

Students will be able to solve the task correctly using one of a number of viable strategies and realize that there are several different and correct ways to solve the task.

Possible Goals

- Students will recognize that the relationship between quantities is multiplicative not additive – that the 2 quantities (leaves and caterpillars) need to grow at a constant rate.

- Student will recognize that there are three related strategies for solving the task – unit rate, scale factor and scaling up.

0₂. Selecting a Task

- **It involves:**
  - Identifying a mathematical task that is aligned with the lesson goals
  - Making sure the task is rich enough to support a discussion (i.e., a cognitively challenging mathematical task)
- **It is supported by:**
  - Setting a clear and explicit goal for learning
  - Using the **Task Analysis Guide** which provides a list of characteristics of tasks at different levels of cognitive demand
  - Working in collaboration with colleagues

The Task Analysis Guide

Lower-Level Demands	Higher-Level Demands
<u>Memorization</u> • involve either reproducing previously learned facts, rules, formulae or definitions OR committing facts, rules, formulae or definitions to memory. • cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure. • are not ambiguous. Such tasks involve exact reproduction of previously-seen material and what is to be reproduced is clearly and directly stated. • have no connection to the concepts or meaning that underlie the facts, rules, formulae or definitions being learned or reproduced.	<u>Procedures With Connections</u> • focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas. • suggest pathways to follow (explicitly or implicitly) that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts. • usually are represented in multiple ways (e.g., visual diagrams, manipulatives, symbols, problem situations). Making connections among multiple representations helps to develop meaning. • require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with the conceptual ideas that underlie the procedures in order to successfully complete the task and develop understanding.
<u>Procedures Without Connections</u> • are algorithmic. Use of the procedure is either specifically called for or its use is evident based on prior instruction, experience, or` placement of the task. • require limited cognitive demand for successful completion. There is little ambiguity about what needs to be done and how to do it. • have no connection to the concepts or meaning that underlie the procedure being used. • are focused on producing correct answers rather than developing mathematical understanding. • require no explanations or explanations that focuses solely on describing the procedure that was used.	<u>Doing Mathematics</u> • require complex and non-algorithmic thinking (i.e., there is not a predictable, well-rehearsed approach or pathway explicitly suggested by the task, task instructions, or a worked-out example). • require students to explore and understand the nature of mathematical concepts, processes, or relationships. • demand self-monitoring or self-regulation of one's own cognitive processes. • require students to access relevant knowledge and experiences and make appropriate use of them in working through the task. • require students to analyze the task and actively examine task constraints that may limit possible solution strategies and solutions. • require considerable cognitive effort and may involve some level of anxiety for the student due to the unpredictable nature of the solution process required.

[redacted]

*These characteristics are derived from the work of Doyle on academic tasks (1988), Resnick on high-level thinking skills (1987), and from the examination and categorization of hundreds of tasks used in QUASAR classrooms (Stein, Grover, & Henningsen, 1996; Stein, Lane, and Silver, 1996).

Stein, Smith, Henningsen, & Silver, 2000, p.16

David Crane's Task

A fourth-grade class needs five leaves each day to feed its 2 caterpillars. How many leaves would the students need each day for 12 caterpillars?

Use drawings, words, or numbers to show how you got your answer.

1. Anticipating

likely student responses to mathematical problems

- **It involves considering:**
  - The array of strategies that students might use to approach or solve a challenging mathematical task
  - How to respond to what students produce
  - Which strategies will be most useful in addressing the mathematics to be learned

- **It is supported by:**
  - Doing the problem in as many ways as possible
  - Discussing the problem with other teachers
  - Drawing on relevant research
  - Documenting student responses year to year

Leaves and Caterpillar: Anticipated Solutions

- **Unit Rate**--Find the number of leaves eaten by one caterpillar and multiply by 12 or add the amount for one 12 times
- **Scale Factor**--Find that the number of caterpillars (12) is 6 times the original amount (2) so the number of leaves must be 6 times the original amount (5)
- **Scaling Up**--Increasing the number of leaves and caterpillars by continuing to add 5 to the leaves and 2 to the caterpillar until you reach the desired number of caterpillars (12)
- **Additive**--Find that the number of caterpillars has increased by 10 (2 + 10 = 12) so the number of leaves must also increase by 10 (5 + 10 = 15)

Leaves and Caterpillar: Incorrect Additive Strategy

Missy and Kate's Solution

They added 10 caterpillars, and so I added 10 leaves.

2 caterpillars $\xrightarrow[+10]{}$ *12 caterpillars*

5 leaves $\xrightarrow[+10]{}$ *15 leaves*

2. Monitoring

students' actual responses during independent work

- **It involves:**
  - Circulating while students work on the problem and watching and listening
  - Recording interpretations, strategies, and points of confusion
  - Asking questions to get students back "on track" or to advance their understanding

- **It is supported by:**
  - Anticipating student responses beforehand
  - Carefully listening and asking probing questions
  - Using recording tools

Monitoring Tool

Strategy	Who and What	Order

Monitoring Tool

Strategy	Who and What	Order

List the different solution paths you anticipated

Monitoring Tool

Strategy	Who and What	Order
Unit Rate--Find the number of leaves eaten by one caterpillar and multiply by 12 or add the amount for one 12 times		
Scale Factor--Find that the number of caterpillars (12) is 6 times the original amount (2) so the number of leaves (30) must be 6 times the original amount (5)		
Scaling Up--Increasing the number of leaves and caterpillars by continuing to add 5 to the leaves and 2 to the caterpillar until you reach the desired number of caterpillars		
Additive--Find that the number of caterpillars has increased by 10 (2 + 10 = 12) so the number of leaves must also increase by 10 (5 + 10 = 15)		
OTHER		

Monitoring Tool

Strategy	Who and What	Order
Unit Rate--Find the number of leaves eaten by one caterpillar and multiply by 12 or add the amount for one 12 times		
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OTHER		

Make note of which students produced which solutions and what you might want to highlight

Monitoring Tool

Strategy	Who and What	Order
Unit Rate--Find the number of leaves eaten by one caterpillar and multiply by 12 or add the amount for one 12 times	Janine (number sentence) Kyra (picture)	
Scale Factor--Find that the number of caterpillars (12) is 6 times the original amount (2) so the number of leaves (30) must be 6 times the original amount (5)	Jason	
Scaling Up--Increasing the number of leaves and caterpillars by continuing to add 5 to the leaves and 2 to the caterpillar until you reach the desired number of caterpillars	Jamal (table) *Martin and Melissa did sets of leaves and caterpillars*	
Additive--Find that the number of caterpillars has increased by 10 (2 + 10 = 12) so the number of leaves must also increase by 10 (5 + 10 = 15)	Missy and Kate	
OTHER—Multiplied leaves and caterpillars	Darnell and Marcus	

3. Selecting
student responses to feature during discussion

- **It involves:**
  - Choosing particular students to present because of the mathematics available in their responses
  - Making sure that over time all students are seen as authors of mathematical ideas and have the opportunity to demonstrate competence
  - Gaining some control over the content of the discussion (no more “who wants to present next?”)
- **It is supported by:**
  - Anticipating and monitoring
  - Planning in advance which types of responses to select

Mr. Crane’s Goals

- Students will recognize that the relationship between quantities is multiplicative not additive – that the 2 quantities (leaves and caterpillars) need to grow at a constant rate.

- Student will recognize that there are three related strategies for solving the task – unit rate, scale factor and scaling up.

Mr. Crane’s Goals

- Students will recognize that the relationship between quantities is multiplicative not additive – that the 2 quantities (leaves and caterpillars) need to grow at a constant rate.
  - **Need to show constant rate of change**
  - **Need to emphasize multiplication**

- Student will recognize that there are three related strategies for solving the task – unit rate, scale factor and scaling up.
  - **Need to show solutions that involve each of the strategies**

Monitoring Tool

Strategy	Who and What	Order
Unit Rate--Find the number of leaves eaten by one caterpillar and multiply by 12 or add the amount for one 12 times	Janine (number sentence) - shows multiplication Kyra (picture)	Need for goal 2
Scale Factor--Find that the number of caterpillars (12) is 6 times the original amount (2) so the number of leaves (30) must be 6 times the original amount (5)	Jason - shows multiplication	Need for goal 2
Scaling Up--Increasing the number of leaves and caterpillars by continuing to add 5 to the leaves and 2 to the caterpillar until you reach the desired number of caterpillars	Jamal (table) – shows relationship between leaves and caterpillars *Martin and Melissa did sets of leaves and caterpillars – all show 2 for 5*	Need for goal 2
Additive--Find that the number of caterpillars has increased by 10 (2 + 10 = 12) so the number of leaves must also increase by 10 (5 +	Missy and Kate	

4. Sequencing
student responses during the discussion

- **It involves:**
  - Purposefully ordering presentations so as to make the mathematics accessible to all students
  - Building a mathematically coherent story line

- **It is supported by:**
  - Anticipating, monitoring, and selecting
  - During anticipation work, considering how possible student responses are mathematically related

Monitoring Tool

Strategy	Who and What	Order
Unit Rate--Find the number of leaves eaten by one caterpillar and multiply by 12 or add the amount for one 12 times	Janine (picture and number sentence) Kyra (picture)	
Scale Factor--Find that the number of caterpillars (12) is 6 times the original amount (2) so the number of leaves (30) must be 6 times the original amount (5)	Jason	
Scaling Up--Increasing the number of leaves and caterpillars by continuing to add 5 to the leaves and 2 to the caterpillar until you reach the desired number of caterpillars	Jamal (table) *Martin and Melissa did sets of leaves and caterpillars*	
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OTHER—Multiplied leaves and caterpillars	Darnell and Marcus	

Indicate the order in which students will share

Monitoring Tool

Strategy	Who and What	Order
Unit Rate--Find the number of leaves eaten by one caterpillar and multiply by 12 or add the amount for one 12 times	Janine (picture and number sentence) Kyra (picture)	3 (Janine)
Scale Factor--Find that the number of caterpillars (12) is 6 times the original amount (2) so the number of leaves (30) must be 6 times the original amount (5)	Jason	4 (Jason)
Scaling Up--Increasing the number of leaves and caterpillars by continuing to add 5 to the leaves and 2 to the caterpillar until you reach the desired number of caterpillars	Jamal (table) *Martin and Melissa did sets of leaves and caterpillars*	2 (Jamal) 1 (Martin)
Additive--Find that the number of caterpillars has increased by 10 (2 + 10 = 12) so the number of leaves must also increase by 10 (5 + 10 = 15)	Missy and Kate	
OTHER—Multiplied leaves and caterpillars	Darnell and Marcus	

Leaves and Caterpillar Vignette

Possible Sequencing:

1. Martin – picture (scaling up)
2. Jamal – table (scaling up)
3. Janine -- picture/written explanation (unit rate)
4. Jason -- written explanation (scale factor)

5. Connecting

student responses during the discussion

- **It involves:**
  - Encouraging students to make mathematical connections between different student responses
  - Making the key mathematical ideas that are the focus of the lesson salient

- **It is supported by:**
  - Anticipating, monitoring, selecting, and sequencing
  - During planning, considering how students might be prompted to recognize mathematical relationships between responses

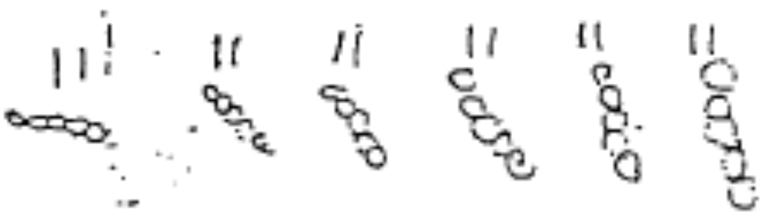
Leaves and Caterpillar Vignette

Possible Connections:

1. Martin – picture (scaling up)
2. Jamal – table (scaling up)
3. Janine -- picture/written explanation (unit rate)
4. Jason -- written explanation (scale factor)

Leaves and Caterpillar Vignette

1. Martin – picture (scaling up)



2. Jamal – table (scaling up)

leaves	5	10	15	20	25	30	
caterpillars	2	4	6	8	10	12	

3. Janine -- picture/written explanation (unit rate)

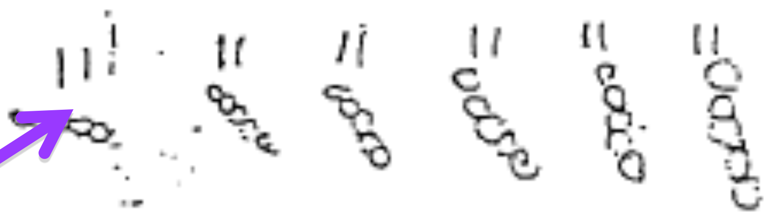
if each of the caterpillars need 2 ½ leaves a day then you just X's 2½ x's 12 = 30.

4. Jason -- written explanation (scale factor)

If it takes 5 leafs for two caterpillars, you just count by twos, until you come to half of 12. The number is six, and then you multiply 5 X 6, and it equals 30.

Leaves and Caterpillar Vignette

1. Martin – picture (scaling up)



2. Jamal – table (scaling up)

leaves	5	10	15	20	25	30
caterpillars	2	4	6	8	10	12

How is Martin's picture related to Jamal's table?

3. Janine -- picture/written explanation (unit rate)

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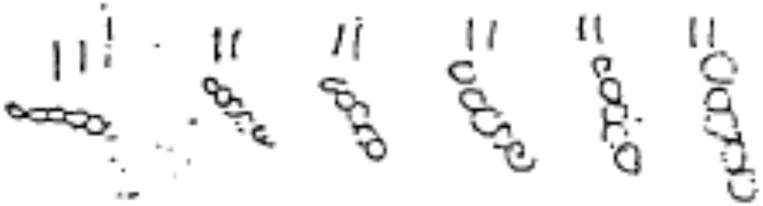
if each of the caterpillars need 2 1/2 leaves a day then you just X's 2 1/2 x's 12 = 30.

4. Jason -- written explanation (scale factor)

If it takes 5 leafs for two caterpillars, you just count by twos, until you come to half of 12. The number is six, and then you multiply 5 X 6, and it equals 30.

Leaves and Caterpillar Vignette

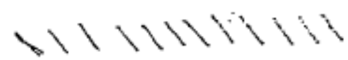
1. Martin – picture (scaling up)



2. Jamal – table (scaling up)

leaves	5	10	15	20	25	30
caterpillars	2	4	6	8	10	12

3. Janine -- picture/written explanation (unit rate)



if each of the caterpillars need 2 ½ leaves a day then you just X's 2½ x's 12 = 30.

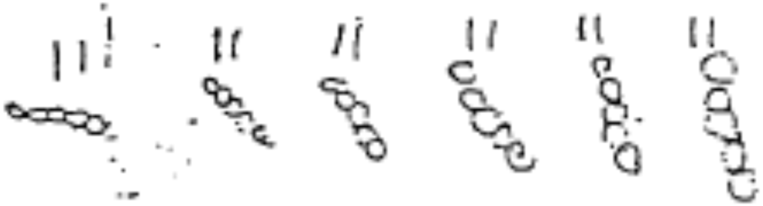
4. Jason – ... explana... factor)

If it takes 5 ... caterpillars, you just count by twos, until you come to half of 12. The number is six, and then you multiply 5X6, and it equals 30.

Where do you see the unit rate of 2 ½ in Jamal's table?

Leaves and Caterpillar Vignette

1. Martin – picture (scaling up)



2. Jamal – table (scaling up)

leaves	5	10	15	20	25	30
caterpillar	2	4	6	8	10	12

3. Janine -- picture/written explanation (unit rate)

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if each of the caterpillars need 2 1/2 leaves a day then you just X's 2 1/2 x's 12 = 30.

4. Jason -- written explanation (scale factor)

If it takes 5 leafs for two caterpillars, you just count by twos, until you come to half of 12. The number is six, and then you multiply 5 X 6, and it equals 30.

Where do you see the scale factor of 6 in the other solutions?

Overview

- Analyze an instructional episode that will provide a basis for talking about classroom discussions
- Describe 5+ practices that you can learn in order to facilitate discussions more effectively and ground each in the instructional episode
- Discuss how the 5 practices could help improve teaching

Why These Five Practices Likely to Help

- **Provides teachers with more control**
  - Over the content that is discussed
  - Over teaching moves: not everything improvisation

- **Provides teachers with more time**
  - To diagnose students' thinking
  - To plan questions and other instructional moves

- **Provides a reliable process for teachers to gradually improve their lessons over time**

Why These Five Practices Likely to Help

- **Honors students' thinking while guiding it in productive, disciplinary directions** (Ball, 1993; Engle & Conant, 2002)
  - Key is to support students' disciplinary authority while simultaneously holding them accountable to discipline
  - Guidance done mostly 'under the radar' so doesn't impinge on students' growing mathematical authority
  - At same time, students led to identify problems with their approaches, better understand sophisticated ones, and make mathematical generalizations
  - This fosters students' accountability to the discipline

Consider….

- What relationship do you see between the 5+ practices and the mathematical practices in the CCSSM?

Consider….

- What would you “look for” in classrooms to see if the 5 practices are being used?

Resources Related to the Five Practices

- Stein, M.K., Engle, R.A., Smith, M.S., & Hughes, E.K. (2008).Orchestrating productive mathematical discussions: Helping teachers learn to better incorporate student thinking. *Mathematical Thinking and Learning, 10*, 313-340.

- Smith, M.S., Hughes, E.K., & Engle, R.A., & Stein, M.K. (2009). Orchestrating discussions. *Mathematics Teaching in the Middle School, 14* (9), 549-556.

Resources Related to the Five Practices

- Smith, M.S., & Stein, M.K. (2011). *5 Practices for Orchestrating Productive Mathematics Discussions*. Reston, VA: National Council of Teachers of Mathematics.

For additional information, you can contact me at

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