Mathematics | High School—Modeling

Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. Quantities and their relationships in physical, economic, public policy, social, and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data.

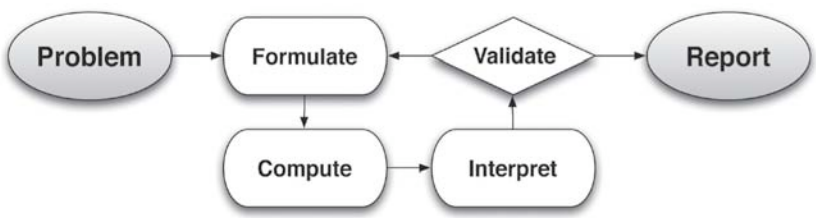
A model can be very simple, such as writing total cost as a product of unit price and number bought, or using a geometric shape to describe a physical object like a coin. Even such simple models involve making choices. It is up to us whether to model a coin as a three-dimensional cylinder, or whether a two-dimensional disk works well enough for our purposes. Other situations—modeling a delivery route, a production schedule, or a comparison of loan amortizations—need more elaborate models that use other tools from the mathematical sciences. Real-world situations are not organized and labeled for analysis; formulating tractable models, representing such models, and analyzing them is appropriately a creative process.
Like every such process, this depends on acquired expertise as well as creativity.

Some examples of such situations might include:

- Estimating how much water and food is needed for emergency relief in a devastated city of 3 million people, and how it might be distributed.
- Planning a table tennis tournament for 7 players at a club with 4 tables, where each player plays against each other player.
- Designing the layout of the stalls in a school fair so as to raise as much money as possible.
- Analyzing stopping distance for a car.
- Modeling savings account balance, bacterial colony growth, or investment growth.
- Engaging in critical path analysis, e.g., applied to turnaround of an aircraft at an airport.
- Analyzing risk in situations such as extreme sports, pandemics, and terrorism.
- Relating population statistics to individual predictions.

In situations like these, the models devised depend on a number of factors: How precise an answer do we want or need? What aspects of the situation do we most need to understand, control, or optimize? What resources of time and tools do we have? The range of models that we can create and analyze is also constrained by the limitations of our mathematical, statistical, and technical skills, and our ability to recognize significant variables and relationships among them. Diagrams of various kinds, spreadsheets and other technology, and algebra are powerful tools for understanding and solving problems drawn from different types of real-world situations.

One of the insights provided by mathematical modeling is that essentially the same mathematical or statistical structure can sometimes model seemingly different situations. Models can also shed light on the mathematical structures themselves, for example, as when a model of bacterial growth makes more vivid the explosive growth of the exponential function.



The basic modeling cycle is summarized in the diagram. It involves (1) identifying variables in the situation and selecting those that represent essential features, (2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables, (3) analyzing and performing operations on these relationships to draw conclusions, (4) interpreting the results of the mathematics in terms of the original situation, (5) validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable, (6) reporting on the conclusions and the reasoning behind them. Choices, assumptions, and approximations are present throughout this cycle.

In descriptive modeling, a model simply describes the phenomena or summarizes them in a compact form. Graphs of observations are a familiar descriptive model— for example, graphs of global temperature and atmospheric CO2 over time.

Analytic modeling seeks to explain data on the basis of deeper theoretical ideas, albeit with parameters that are empirically based; for example, exponential growth of bacterial colonies (until cut-off mechanisms such as pollution or starvation intervene) follows from a constant reproduction rate. Functions are an important tool for analyzing such problems.

Graphing utilities, spreadsheets, computer algebra systems, and dynamic geometry software are powerful tools that can be used to model purely mathematical phenomena (e.g., the behavior of polynomials) as well as physical phenomena.

Modeling Standards *Modeling is best interpreted not as a collection of isolated topics but rather in relation to other standards. Making mathematical models is a Standard for Mathematical Practice, and specific modeling standards appear throughout the high school standards indicated by a star symbol* (★).

The Modeling Process

The process of beginning with a situation and gaining understanding about that situation is generally referred to as "modeling". If the understanding comes about through the use of mathematics, the process is known as **mathematical modeling**.

Step 1. Identify a situation.

Notice something that you wish to understand, and pose a well-defined question indicating exactly what you wish to know.

Step 2. Simplify the situation.

List the key features (and relationships among those features) that you wish to include for consideration. These are the assumptions on which your model will rest. Also note features and relationships you choose to ignore for now.

Step 3. Build the model. Solve the problem.

Interpret in mathematical terms the features and relationships you have chosen. (Define variables, write equations, draw shapes, measure objects, calculate probabilities, gather data and organize into tables, make graphs, etc.). *That* is the model. Then, apply the model and solve the problem. (Solve the equation, draw inferences from patterns in the data, compare results to a standard result, etc.)

Step 4. Evaluate and revise the model.

Go back to the original situation and see if results of mathematical work make sense. If so, use the model until new information becomes available or assumptions change. If not, reconsider the assumptions you made in step 2 and revise them to be more realistic.

A first look at modeling…

A jigsaw puzzle company wants to fill this jar with M&M candies so that a photograph can be taken with the jar looking more or less full (a handful shy will not make any difference). How many M&M's should they buy? You have a ruler, a few representative M&M's (don't eat them until you are done), and 15 minutes to complete this project.

Some ideas to consider:

1. You don't have to come up with the perfect answer immediately. You can start *simply* and revise your answer by using more complexity given time.
2. Since the exact number isn't important, how would you *approximate* the number?

Be sure to carefully list what assumptions you are making about the physical scenario.

Are you done? Can you make a general form of a solution using variables? How general can you make it?
A first look at modeling…

Some Additional Problems

McDonald's Claim

Wikipedia reports that 8% of all Americans eat at McDonald's everyday. There are 310 million Americans and 12,800 McDonald's. Do you believe the Wikipedia report to be rue? Create a mathematical argument to justify your position.

Elevator Problem

Suppose a building has 5 floors (1-5) which are occupied by offices. The ground floor (0) is not used for business purposes. Each floor has 80 people working on it, and there are 4 elevators available. Each elevator can hold 10 people at one time. The elevators take 3 seconds to travel between floors and average 22 seconds on each floor when someone enters or exits. If all of the people arrive at work at about the same time and enter the elevators on the ground floor, how should the elevators be used to get the people to their offices as quickly as possible?

Pasture Land

A rancher has a prize bull and some cows on his ranch. He has a large area for pasture that includes a stream running along one edge. He must divide the pasture into two regions, one region large enough for the cows and the other, smaller region to hold the bull. The bull's pasture must be at least 1,000 square meters (grazing) and the cow pasture must be at least 10,000 square meters to provide grazing for the cows. The shape of the pasture is basically a rectangle 120 meters by 150 meters. The river runs all the way along the 120 meter side. Fencing costs $5 per meter and each fence post costs $10. Any straight edge of fence requires a post every 20 meters and any curved length of fence requires a post every 10 meters. Help the rancher minimize his total cost of fencing.

Faculty

When confronted with a rise of 142 students in a school of 480, and a capacity of 7 new teachers, in what departments should the new teachers be placed? Placing the new teachers should maintain the ideal student-teacher ratio. The current makeup of the faculty and student enrollment in each department is:
Art 1:99 Biology 4:319 Chemistry 3:294 English 5:480 French 1:122 German 1:51 Spanish 1:110 Mathematics 6:613 Music 1:95 Physics 3:291 Social Studies 4:363

A Country and Its Food Supply

The population of a country is initially 2 million people and is increasing at 4% per year. The country's annual food supply is initially adequate for 4 million people and is increasing at a constant rate adequate for an additional 0.5 million people per year.
a. Based on these assumptions, in approximately what year will this country first experience shortages of food?
b. If the country doubled its initial food supply, would shortages still occur? If so, when? (Assume the other conditions do not change.)
c. If the country doubled the rate at which its food supply increases, in addition to doubling its initial food supply, would shortages still occur? If so, when? (Again, assume the other conditions do not change.)