Guide to Scoring Rubrics

What is a scoring rubric?

How do you know if your students have learned something you've taught in the classroom? Evaluating the learning process is no simple task. Since learning is a dynamic and complex process, teachers need a diverse set of tools for measuring the progress of his/her students. One of those tools is the scoring rubric.

A scoring rubric is a standard of performance for a defined population. It is a pre-determined set of goals and objectives on which to base an evaluation. In the Higher Education Report, **S.M. Brookhart** describes a scoring rubric as, "Descriptive scoring schemes that are developed by teachers or other evaluators to guide the analysis of the products or processes of students' efforts."

This article will explore in depth the different types of scoring rubrics, how to make one yourself, as well as an analysis into how scoring rubrics enhance learning.

Types of scoring rubrics

Despite the overwhelming number of scoring rubrics you can find on the Internet and in various textbooks and curriculum guides, most rubrics fall into one of two categories: Analytic or holistic scoring rubrics.

Analytic scoring rubrics

Analytic rubrics attempt to break down the final product or goal into measurable components and parts. In other words, your student has a project or assignment and you use an analytic scoring rubric to evaluate all the pieces of the project. Analytic rubrics typically use numbers to measure quality. Let's take the example below.

Student Assignment: Write a one-page paper on your summer vacation.

The rubric might break down the evaluation process into three parts- content of the paper, grammar and mechanics, and organization of ideas. For each of these components, numbers would be assigned.

(1) Needs improvement, (2) Developing, (3) Goal, (4) Above average, (5) Excellent

The rubric also explains what exactly each of those numbers mean. So a student might have a score like this:

Content (3) – Ideas were developed and thought out. Examples were given.
Grammar (4) – The paper was free of all spelling and grammar errors. There were only a few awkward sentences.
Organization (2) – Each idea was not separated out into paragraphs. Author jumped around and confused the reader.

With an analytic scoring rubric, the student and teacher can see more clearly what areas need work and what areas are mastered. It is far more descriptive than a simple A, B, or C grade.

Holistic scoring rubrics

Whereas analytic rubrics break down the assignment into measurable pieces, a holistic scoring rubric evaluates the work as a whole. In the above example, a holistic rubric would look like this:

Student Assignment: Write a one-page paper on your summer vacation.

(1) Needs improvement: The story is not clearly organized, grammar errors make it difficult to understand, and content is lacking.

(2) Developing: The student has a grasp on the assignment but needs to spend more time organizing thoughts, adding details, and fixing errors.

(3) Goal: The student has completely the paper using good content, correct, grammar, and a logical organization of ideas.

(4) Above average: The story is full of great content, organized well, and free from spelling and grammar errors.

(5) Excellent: The student went above and beyond, adding rich detail to his/her story. The content is interesting and organized well. Thoughts are well described. Grammar and mechanics are flawless.

With this rubric, the piece is evaluated as a whole.

General or task-specific?

Rubrics can be either. General rubrics are used across multiple assignments. Once you have developed a general rubric, you can use it to measure different subjects and lessons. Task-specific rubrics are designed to evaluate one specific assignment. Using these guidelines, you can categorize your rubrics into one of the following categories:

General holistic scoring rubric

General analytic scoring rubric

Task-specific analytic scoring rubric

Task-specific holistic scoring rubric

General holistic rubrics have advantages and disadvantages. If you spend the time to create a solid scoring rubric, you won't have to do it again. Students will quickly grasp the "meaning" of each number- therefore understanding what needs improving from assignment to assignment. The value of each number is clear. The disadvantage to this type of rubric is that different subjects may need more specific scoring instructions. With the same rubric used over and over again, your or your students might get stuck in a rut – always using the same score.

General analytic scoring rubrics are difficult to create. Since an analytic rubric is designed to break an assignment into pieces, the best bet is to create a general analytic rubric for a particular subject (like one for writing, one for math, one for reading, etc.). Each subject has similar "measurables" – something that would be difficult to create across different disciplines.

Task-specific analytic scoring rubrics are the most comprehensive and detailed. While they provide a great source of feedback to the student and teacher, it does require more work upfront to create. Creating a task-specific analytic rubric for each assignment would be tremendously tedious. Save these types of rubrics for projects that are large and need to be broken down into parts and pieces for your students to manage and understand.

Task-specific holistic rubrics are like the "balanced" middle of the road rubric. They are designed for a particular assignment, but evaluate it as a whole rather than in parts.

Creating a scoring rubric

Why is it important to create scoring rubrics for your students? Well for one, it helps to spell out clearly what you expect from them in terms of quality, content, and effort. It gives you an objective criterion on which to base a grade, eliminating a lot of the "It's not fair!" mentality that can creep in when grades seem arbitrary. It allows your students the opportunity to understand more comprehensively your expectations of performance. A scoring rubric can also be used for peer-to-peer evaluation. This is another way to engage your students in the learning process.

1. Decide what kind of rubric you are going to make- general or task specific, and then analytic or holistic.
2. Use a Word processing software or Excel to make a chart.
3. If you are creating an analytic scoring rubric, divide the project or assignment up into parts (for example, a math project might have the categories – creativity, understanding of mathematical concepts, correct answers, presentation, effort, etc.).
4. Place these categories in one column down the left side of the table or chart.
5. Create a scoring method. You can use numbers (i.e. 1-5) and attach words to each number (like 1 is poor, 2 is below average, 3 is average, 4 is above average, and 5 is excellent). If it is a task-specific analytic rubric, you can be even more descriptive.
6. Put these scores along the top of the chart in one row. Each score should represent a column.
7. Now you have to write up a short blurb for each category and score. Here is an example of a task-specific analytic scoring rubric for a ***math*** project.

Category	**1 – poor**	**2 – below average**	**3 – average**	**4 – above average**	**5 - excellent**
Effort	Student's work shows little preparation, creativity or effort. Lots of errors and sloppy handwriting.	Student put for minimal effort. Has a few errors and could have added more to the presentation.	Student gave effort to the project. Met all the expectations. Didn't go above and beyond.	Student spent a lot of time working to make sure the presentation was well done. Got help and asked for feedback.	Student went above and beyond the assignment. Did extra research and work.
Understanding of concepts	Didn't incorporate concepts into project. Misunderstood the ideas and principles.	Understood a few of the concepts, but still left out pieces and parts of the assignment.	Student understood concepts and completed all the tasks in the assignment.	Student understood the concepts and did more than what was expected of him/her.	Student mastered the concepts and even added more to the principles.
Correct answers to problems	Most or all of the answers to each problem were incorrect.	Half of the problems were incorrect.	Student got most of the problems correct with only a few errors.	Student got every problem correct.	Student got every problem correct, including the bonus work.
Presentation	Presentation was rushed, sloppy, and too short. Lacked effort and/or visual tools.	Presentation was short and lacking creativity. Some visuals were used.	Presentation was correct length. Student used visuals.	Presentation was well done, with visuals, interaction with the class, and comprehensive.	Presentation was creative, excellently done, using visuals and props.

The student should be given the scoring rubric before the project begins. This way, he/she understands exactly what you are grading on and how you will assess performance. Once you've graded the presentation with the rubric, you can add up the scores and take the average. When using a 1-5 model, it's easy to assign 1=F, 2=D, 3=C, 4=B, 5=A.

You can also leave an extra column to write in comments about each category. Whenever possible, write criterion that are measurable. Use specifics. For general rubrics, this is a bit more challenging, but you can get some idea by perusing online rubrics to see what kind of language other educators use.

Using descriptive gradations

The example above gives you some generic terms to use (like poor, average, etc.), but depending on the task, other words might work better to describe your expectations and criteria. Here are some options to try:

1. Beginning, developing, accomplished, exemplary
2. No, maybe, yes
3. Missing, unclear, clear, thorough
4. Below expectations, basic, proficient, outstanding
5. Never, rarely, sometimes, often, always
6. Novice, apprentice, proficient, master
7. Lead, bronze, silver, gold
8. Byte, kilobyte, megabyte, gigabyte
9. Adagio, andante, moderato, allegro

Using your students to create rubrics

It is crucial that you use language your students can understand. For younger children, you might even use images (of a smiley to sad face for example) to help them understand the expectations. When creating a task-specific analytic rubric, start by drawing the rubric on a whiteboard or poster and have them come up with the language to express what is required.

This writing rubric below is a simplified example that a teacher might use for an elementary assignment.

Criteria	Complete	Unsure	Missing
Beginning	I had a beginning to my story	I don’t know if I had a beginning.	My story started without a beginning.
Character	I mentioned my character’s name and described him/her.	I mentioned my character’s name but didn’t describe him/her	I forgot to introduce my character.
Problem	I wrote about the character’s problem.	I don’t know if the character has a story problem.	The character doesn’t have a story problem.
Ending	I wrote about what happened to the character after the problem.	I wrote an ending but didn’t tell people what happened to the character.	I didn’t write an ending.

Weighted rubrics

Sometimes you want one part of the rubric to count more than others. A simple way to do this is to assign percentages to each category. In the example below (the math scoring rubric), the understanding of concepts and the correct answers categories are going to weigh more heavily. For purposes of our example, let's assign criterion two and three 40% of the project.

Criterion	**1 – poor**	**2 – below average**	**3 – average**	**4 – above average**	**5 - excellent**
1. Effort **10%**	Student's work shows little preparation, creativity or effort. Lots of errors and sloppy handwriting.	Student put for minimal effort. Has a few errors and could have added more to the presentation.	Student gave effort to the project. Met all the expectations. Didn't go above and beyond.	Student spent a lot of time working to make sure the presentation was well done. Got help and asked for feedback.	Student went above and beyond the assignment. Did extra research and work.
2. Understanding of concepts **40%**	Didn't incorporate concepts into project. Misunderstood the ideas and principles.	Understood a few of the concepts, but still left out pieces and parts of the assignment.	Student understood concepts and completed all the tasks in the assignment.	Student understood the concepts and did more than what was expected of him/her.	Student mastered the concepts and even added more to the principles.
3. Correct answers to problems **40%**	Most or all of the answers to each problem were incorrect.	Half of the problems were incorrect.	Student got most of the problems correct with only a few errors.	Student got every problem correct.	Student got every problem correct, including the bonus work.
4. Presentation **10%**	Presentation was rushed, sloppy, and too short. Lacked effort and/or visual tools.	Presentation was short and lacking creativity. Some visuals were used.	Presentation was correct length. Student used visuals.	Presentation was well done, with visuals, interaction with the class, and comprehensive.	Presentation was creative, excellently done, using visuals and props.

So in this case, the student got a 2 in criteria 1, a 4 in criteria 2, a 3 in criteria 3, and a 2 in criteria 4. If you did not weight the grade, the average score would equal 2.2 or a D. However with a weighted rubric, the most important parts of the grade should account for more. Out of a possible 100%, each number should be counted according to the percentage given. Your formula would look like this:

2 + 4 + 4 + 4 + 4 + 3 + 3 + 3 + 3 + 2 = 32 \ 10 = 3.2 or a C.

(Criterion 2 and 3 are each counted four times, and 1 and 4 are counted once – equally ten points or 1.0)

Looking at this rubric, it would seem that a C is a better (or more accurate) grade for this student. They completed the problems and understood the concepts, but didn't spend a lot of time and energy in the presentation part of it.

Sites for scoring rubric resources

If you are short on time or simply need a little help getting started, the following list will help you find excellent already-made scoring rubrics. There are also sites that can help you create them as well!

Rubric generators

1. **iRubric** – Free rubric building tools plus options for analyzing data and sharing rubrics with other teachers around the world.
2. **Teach-nology**- A comprehensive list of rubric building tools arranged by subject.
3. **Digi-tales** – Create a scoring rubric for evaluating media projects.
4. **The Canadian Teacher** – A rubric builder that allows you to build weighted rubrics, checklists, and rating scales.
5. **Rubistar**- Register for an account and have access to a variety of rubric tools, plus the ability to edit, save, and access online.
6. **Scholastic** – A simple and fast rubric tool. Fill in the fields and it will arrange it in a matrix for you.

Premade scoring rubrics

1. **Exemplars** – Standard rubrics for math, science, reading, and writing. They offer some student evaluation rubrics as well.
2. **Teacher Rubrics for Secondary and College** – This website is a list of rubrics that one faculty member has made available for other teachers.
3. **University of Wisconsin** – Rubrics for wikis, web projects, PowerPoint, oral presentations, as well as general subject areas like math and writing.
4. **Teacher planet** – Rubrics are organized by subject and level. They also offer a rubric generator too.
5. **Kathy Schrock** – One of the largest lists of common core rubrics.

Subscription scoring rubric websites

1. **Rubrix** – Designed for school systems and HR professionals. Full set of tools, mobile functions, and more.
2. **rGrade** - Comprehensive assessment management system.

How do scoring rubrics enhance learning?

First and foremost, a scoring rubric makes it easy for your students to understand your expectations as the teacher. When an assignment is given without a rubric, there are a lot of assumptions that can be made about the quality, quantity, and project outcome that can result in rabbit trails and a poor grade. Rubrics spell everything out in an easy digestible format.

1. Rubrics help educators' grade projects fairly.
2. Rubrics speed up the grading process with clearly outlined goals.
3. Rubrics allow the student to use the scoring sheet to grade someone else's work.
4. Rubrics are an easy way for parents to understand the final grade on the assignment.
5. Rubrics help to define the goal and reason for the assignment or project.
6. Rubrics keep students on track during the course of the assignment.
7. Rubrics give more specific feedback so that the student can see where his/her strengths and weaknesses lie.
8. Rubrics are a tool to help the student dig deeper into an assignment.
9. Rubrics are easy to understand and can help give instructions about the project.
10. Rubrics outline various skill sets that students should be aware of during the assignment.
11. Rubrics allow students to check their work throughout the project for instant monitoring and feedback.
12. Rubrics give teachers data for future planning and curriculum design.
13. Rubrics ensure that different teachers will all grade a project using the same criterion and goals.

So are there any disadvantages to scoring rubrics?

Even though rubrics are a great classroom tool, there are a few pitfalls to avoid. For one, scoring rubrics can take a long time to create – especially if they are task-specific and you spend time thinking through each criterion carefully. A teacher's work needs to be balanced between instruction, mentorship, and feedback. Try not to get caught up in creating a custom rubric for every single assignment. Don't be afraid to use rubrics that are already made up for you.

1. Watch out for rubrics that are poorly designed. If the criteria are not thought out well, then your students will be heading in the wrong direction.
2. Too many rubrics can cause creativity to dwindle. If your students are always performing to the written standard, they may be less likely to think outside the box.
3. Rubrics may cause your most intelligent students to underperform. Once in a while, let their imaginations determine how high or far they can go in an assignment. It may be further than you dreamed.
4. Poor descriptions will render a scoring rubric useless. Make your assessments as specific as possible.
5. Rubrics can overwhelm students if the criterion is lengthy. Maybe breaking the project into parts with "mini" rubrics would be more helpful.
6. Some educators say that turning rubric scores into grades is unhelpful. Scoring rubrics should be the extent of the evaluation, not trying to turn it into an A, B, or C.

Ultimately, balance is key. Scoring rubrics are a great asset to both teachers and students, as long as the classroom isn't wholly designed to simply meet a goal. We all know that learning is far more dynamic and creativity than what can fit inside a little box.

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- See more at: http://www.guide2digitallearning.com/teaching_learning/guide_scoring_rubrics#sthash.pDYT1hSC.dpuf

Math Project Rubric

	1	2	3	4
Mathematical Knowledge and Understanding (30%)	The student attempts to apply methods, rules and/ or formulas to instruct their topic.	The student sometimes uses appropriate methods, rules and/ or formulas to instruct their topic.	The student generally uses appropriate methods, rules and/ or formulas to instruct their topic.	The student consistently uses appropriate methods, rules and/ or formulas to instruct their topic.t.
Level of Difficulty (10%)	Topic has been covered in class and not extended.	Topic has been covered in class and their has been an attempt to extend the topic.	Topic has been covered in class and has been successfully extended.	Topic explores beyond material covered in class.
Use of Math Technology (20%)	The student uses the computer (grapher, equation editor) or TI-84 calculator for only routine calculations in their topic.	The student attempts to use a TI-84 calculator or computer (grapher, equation editor) in a manner that could enhance the development of their topic.	The student makes a limited amount of use of a TI-84 calculator or computer (grapher, equation editor) in a manner that enhances the development of their topic.	The student makes full use of a TI-84 calculator or computer (grapher, equation editor) in a manner that enhances the development of their topic.
Communication (20%)	Topic and/or question has not been stated or introduced. The student shows no use of mathematical language and/or forms of mathematical representation (formulas, diagrams, tables, charts, graphs, and models). Reasoning, explanations and conclusion are non-existent. There are no references.	Topic and/or question has been poorly stated or introduced. The student shows basic use of mathematical language and/or forms of mathematical representation (formulas, diagrams, tables, charts, graphs, and models). Reasoning, explanations and conclusions are difficult to follow. References are poorly done.	Topic and/or question has been stated or introduced. The student shows some use of mathematical language and forms of mathematical representation (formulas, diagrams, tables, charts, graphs, and models). Reasoning, explanations and conclusions are logical but not always complete. References are included.	Topic and/or question has been stated or introduced. The student shows good use of mathematical language and forms of mathematical representation (formulas, diagrams, tables, charts, graphs, and models). Reasoning, explanations and conclusions are logical and complete. References are included.
Presentation (20%)	Little effort appears to have been put into the presentation.	Problems with the presentation make it difficult to follow.	The presentation is generally easy to follow and it is obvious that some effort has been made.	The presentation is easy to follow and it is obvious that considerable effort has been made.

Classic Exemplars Rubric

Level	Understanding	Strategies, Reasoning, Procedures	Communication
Novice	• There is no solution, or the solution has no relationship to the task. • Inappropriate concepts are applied and/or procedures are used. • The solution addresses none of the mathematical components presented in the task.	• No evidence of a strategy or procedure, or uses a strategy that does not help solve the problem. • No evidence of mathematical reasoning. • There were so many errors in mathematical procedures that the problem could not be solved.	• There is no explanation of the solution, the explanation cannot be understood or it is unrelated to the problem. • There is no use or inappropriate use of mathematical representations (e.g. figures diagrams, graphs, tables, etc.). • There is no use, or mostly inappropriate use, of mathematical terminology and notation.
Apprentice	• The solution is not complete indicating that parts of the problem are not understood. • The solution addresses some, but not all of the mathematical components presented in the task.	• Uses a strategy that is partially useful, leading some way toward a solution, but not to a full solution of the problem. • Some evidence of mathematical reasoning. • Could not completely carry out mathematical procedures. • Some parts may be correct, but a correct answer is not achieved.	• There is an incomplete explanation; it may not be clearly presented. • There is some use of appropriate mathematical representation. • There is some use of mathematical terminology and notation appropriate of the problem.
Practitioner	• The solution shows that the Student has a broad understanding of the problem and the major concepts necessary for its solution. • The solution addresses all of the mathematical components presented in the task.	• Uses a strategy that leads to a solution of the problem. • Uses effective mathematical reasoning. • Mathematical procedures used. • All parts are correct and a correct answer is achieved.	• There is a clear explanation. • There is appropriate use of accurate mathematical representation. • There is effective use of mathematical terminology and notation.
Expert	• The solution shows a deep understanding of the problem including the ability to identify the appropriate mathematical concepts and the information necessary for its solution. • The solution completely addresses all mathematical components presented in the task. • The solution puts to use the underlying mathematical concepts upon which the task is designed.	• Uses a very efficient and sophisticated strategy leading directly to a solution. • Employs refined and complex reasoning. • Applies procedures accurately to correctly solve the problem and verify the results. • Verifies solution and/or evaluates the reasonableness of the solution. • Makes mathematically relevant observations and/or connections.	• There is a clear, effective explanation detailing how the problem is solved. All of the steps are included so that the reader does not need to infer how and why decisions were made. • Mathematical representation is actively used as a means of communicating ideas related to the solution of the problem. • There is precise and appropriate use of mathematical terminology and notation

Exemplars® K-12 We Set the Standards!

General Scoring Rubric for Written Response Items

Category	Score	Description
No Response	0	Either the work is not attempted (i.e., the paper is blank), or the work is incorrect, irrelevant, or off task. The response may minimally interpret or re-state the problem, but does not go beyond that.
Minimal	1	The response demonstrates only a minimal understanding of the problem posed and a reasonable approach is not suggested. Although there may or may not be some correct mathematical work, the response is incomplete, contains major mathematical errors, or reveals serious flaws in reasoning. Requested examples may be absent or irrelevant.
Partial	2	The response contains evidence of a conceptual understanding of the problem in that a reasonable approach is indicated. However, on the whole, the response is not well developed. Although there may be serious mathematical errors or flaws in reasoning, the response does contain some correct mathematics. Requested examples provided may fail to illustrate the desired conclusions.
Satisfactory	3	The response demonstrates a clear understanding of the problem and provides an acceptable approach. The response also is generally well developed and presented, but contains omissions or minor errors in mathematics. Requested examples provided may not completely illustrate the desired conclusions.
Excellent	4	The response demonstrates a complete understanding of the problem, is correct, and the methods of solution are appropriate and fully developed. The response is logically sound, clearly written, and does not contain any significant errors. Requested examples are well chosen and illustrate the desired conclusions.

EXPLANATORY NOTES

(1) Rubrics for specific items should always be used with this general rubric and the following notes about specific rubrics.

(2) The following excerpt from MDTP Guidelines for The Preparation of Written Response Mathematics Questions provides a context for this general rubric. The statement of the question should be explicit and clear. The extent to which students are to discuss their reasoning and results should be explicit. The extent to which students are to provide examples, counterexamples, or generalizations should also be clearly stated.

(3) Although the categories in the General Scoring Rubric are meant to indicate different levels of understanding and accomplishment, teachers should expect that some student responses may be on the boundary between two categories and may be scored differently by different teachers.

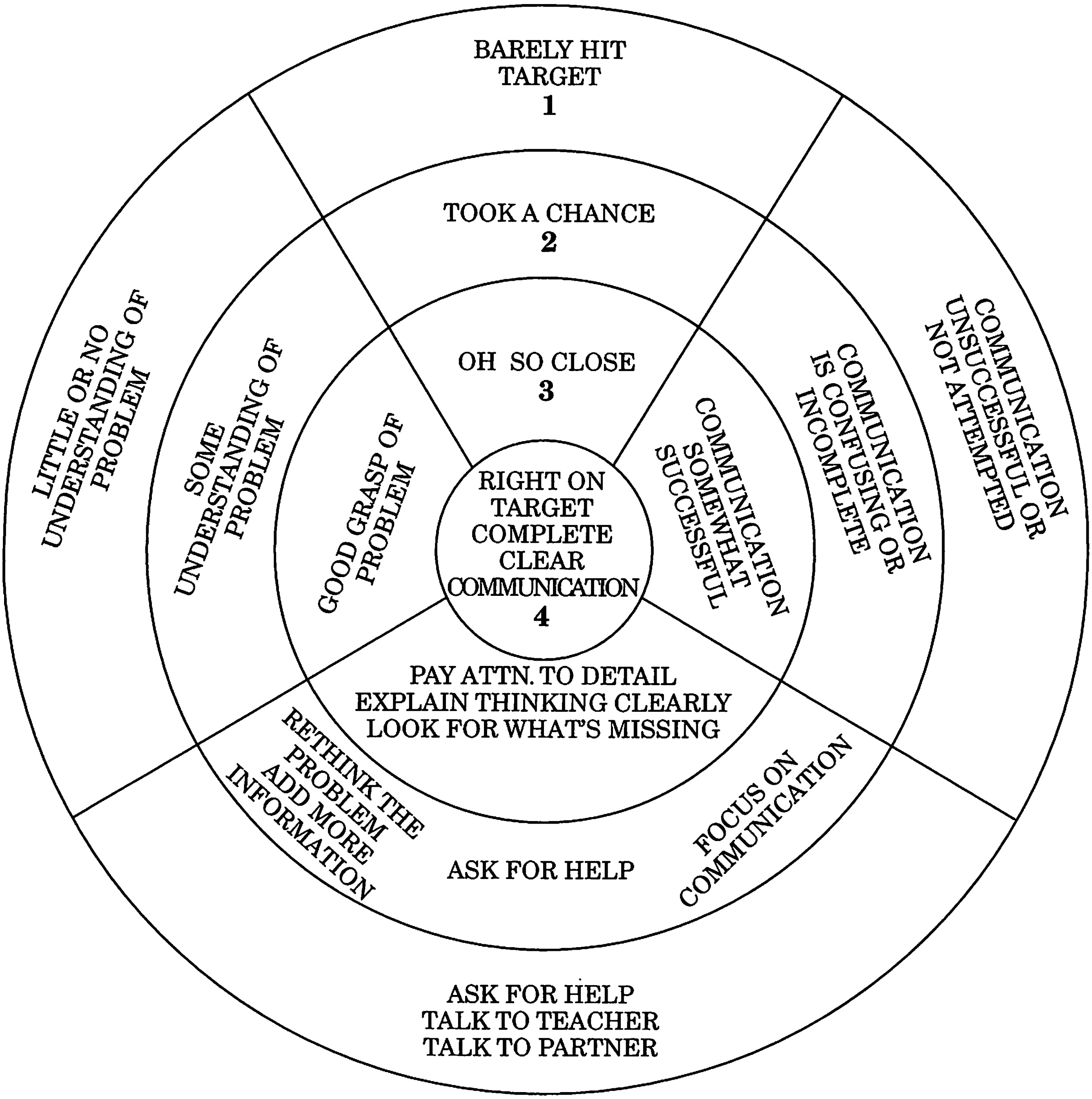
(4) Teachers may wish to designate some outstanding responses in the Excellent category as exemplars.

NOTES EXPLAINING HOW TO USE SPECIFIC ITEM RUBRICS

Scoring of written responses is to be based upon both the correctness of the mathematics and the clarity of the presentation. In scoring, do NOT "mind read" the presenter; instead only grade the presentation. Grade each response on the actual mathematics written and on the quality of the presentation of that mathematics. Unexecuted recipes or prescriptions should receive minimal credit. The specific scoring rubric for an item outlines the mathematical development necessary for the given scores. In addition to the formal mathematics, it is essential that students "show their work" and clearly present their methodology. The evaluation of each response should be based in part upon its organization, completeness, and clarity. A score of 1 or 2 may in some cases be based simply upon the mathematics called for in the rubric. Scores of 3 and 4 require effective presentation as well as appropriate mathematics. The mathematics called for in specific rubrics is necessary, but not sufficient, for these scores.

A Model for Interpreting Scores

AIMING FOR SUCCESS IN PROBLEM SOLVING



Stone Creek School
Irvine Unified

	Emerging (1)	Developing (2)	Proficient (3)	Exemplary (4)
Introduction Key Question: *Does the student's interpretation of the problem accurately reflect the important mathematics in the problem?*	• The data you showed was inaccurate. • You used the wrong information in trying to solve the problem. • You did not state what the problem is. • You did not indicate where you were headed in solving the problem.	• The data you show is accurate, but poorly organized. • You used some but not all of the relevant information from the problem. • You stated what the problem is incorrectly. • You partially indicated where you were headed with your solution.	• Your data is organized and accurate, but includes extraneous information not needed to solve the problem. • You used all relevant information from the problem in your solution. • You stated what part of the problem is correctly, but failed to mention other aspects of the problem. • You indicated where you were headed with your solution.	• The data shown is only the data needed to solve the problem and it is well organized and accurate. • You uncovered hidden or implied information not readily apparent. • You stated what all parts of the problem are correctly. • You indicated the starting and ending points for your solution.

	Emerging (1)	Developing (2)	Proficient (3)	Exemplary (4)
Methods Key Question: *Is there evidence that the student proceeded from a plan, applied appropriate strategies, and followed a logical and verifiable process toward a solution?*	• Your mathematical representations of the problem were incorrect. • Your strategies were not appropriate for the problem. • You didn't seem to know where to begin. • Your reasoning did not support your work. • There was no apparent relationship between your representations and the task. • Your approach to the problem would not lead to a correct solution.	• You used an oversimplified approach to the problem. • You offered little or no explanation of your strategies. • Your choice of forms to represent the problem was inefficient or inaccurate. • Some of your representations accurately depicted aspects of the problem. • You sometimes made leaps in your logic that were hard to follow. • Your process would lead to a partially complete solution.	• You chose appropriate, efficient strategies for solving the problem. • You justified each step of your work. • Your choices of mathematical representations of the problem were appropriate. • The logic of your solution was apparent. • Your process would lead to a complete, correct solution of the problem.	• You chose innovative and insightful strategies for solving the problem. • Your choice of mathematical representations helped clarify the problem's meaning. • You used a sophisticated approach to solve the problem. • You chose mathematical procedures that would lead to an elegant solution.

Mathematics Problem Solving Scoring Guide
Based on a Rubric from Northwest Regional Educational Laboratory

	Emerging (1)	Developing (2)	Proficient (3)	Exemplary (4)
Results Key Question: *Given the approach taken by the student, is the solution performed in an accurate and complete manner?*	• Errors in computation were serious enough to flaw your solution. • Your mathematical representations were inaccurate. • You labeled incorrectly. • Your solution was incorrect. • You gave no evidence of how you arrived at your answer. • There was no apparent logic to your solution.	• You made minor computational errors. • Your representations were essentially correct but not accurately or completely labeled. • Your inefficient choice of procedures impeded your success. • The evidence for your solution was inconsistent or unclear.	• Your computations were essentially accurate. • All visual representations were complete and accurate. • Your solution was essentially correct. • Your work clearly supported your solution.	• All aspects of your solution were completely accurate. • You used multiple representations for verifying your solution. • You showed multiple ways to compute your answer. • You proved that your solution was correct and that your approach was valid.
Discussion Key Question: *Does the student grasp the deeper structure of the problem and see how the process used to solve this problem connects it to other problems or "real-world" applications?*	• You were unable to recognize patterns and relationships. • You found a solution and then stopped. • You found no connections to other disciplines or mathematical concepts.	• You recognized some patterns and relationships. • You found multiple solutions but not all were correct. • Your solution hinted at a connection to an application or another area of mathematics.	• You recognized important patterns and relationships in the problem. • You found multiple solutions using different interpretations of the problem. • You connected your solution process to other problems, areas of mathematics or applications.	• You created a general rule or formula for solving related problems. • You related the underlying structure of the problem to other similar problems. • You noted possible sources of error or ambiguity in the problem. • Your connection to a real-life application was accurate and realistic.

	Emerging (1)	Developing (2)	Proficient (3)	Exemplary (4)
Communication Key Question: *Was I able to easily understand the student's thinking or did I have to make inferences and guesses about what they were trying to do?*	• You had many spelling and/or grammatical errors that detract from your argument. • I couldn't follow your thinking. • Your explanation seemed to ramble. • You gave no explanation for your work. • You did not seem to have a sense of what your audience needed to know. • Your mathematical representations did not help clarify your thinking. • You used mathematical terminology incorrectly.	• You had spelling and/or grammatical errors, but they do not detract from your argument. • Your solution was hard to follow in places. • I had to make inferences about what you meant in places. • You weren't able to sustain your good beginning. • Your explanation was redundant in places. • Your mathematical representations were somewhat helpful in clarifying your thinking. • You used mathematical terminology imprecisely.	• There were no spelling and/or grammatical errors. • I understood what you did and why you did it. • Your solution was well organized and easy to follow. • Your solution flowed logically from one step to the next. • You used an effective format for communicating. • Your mathematical representations helped clarify your solution. • You used mathematical terminology correctly.	• Your explanation was clear and concise. • You communicated concepts with precision. • Your mathematical representations expanded on your solution. • You gave an in-depth explanation of your reasoning. • You used mathematical terminology precisely.

Exemplars® Jigsaw Student Rubric

Level	Problem Solving	Reasoning and Proof	Communication	Connections	Representation
Novice Makes an effort No or little understanding	I did not understand the problem.	My math thinking is not correct.	I used no math language and/or math notation.	I did not notice anything about the problem or the numbers in my work.	I did not use a math representation to help solve the problem and explain my work.
Apprentice Okay, good try Unclear if student understands	I only understand part of the problem. My strategy works for part of the problem.	Some of my math thinking is correct.	I used some math language and/or math notation.	I tried to notice something, but it is not about the math in the problem.	I tried to use a math representation to help solve the problem and explain my work, but it has mistakes in it.
Practitioner Excellent Clear Strong understanding Meets the standard	I understand the problem and my strategy works. My answer is correct.	All of my math thinking is correct.	I used math language and/or math notation accurately throughout my work.	I noticed something about my math work.	I made a math representation to help solve the problem and explain my work, and it is labeled and correct.
Expert Wow, awesome! Exceptional understanding!	I understand the problem. My answer is correct. I used a rule, and/or verified that my strategy is correct.	I showed that I knew more about a math idea that I used in my plan. Or, I explained my rule.	I used a lot of specific math language and/or notation accurately throughout my work.	I noticed something in my work, and used that to extend my answer and/or I showed how this problem is like another problem.	I used another math representation to help solve the problem and explain my work in another way.

www.exemplars.com

Quadratic Equation Math Rubric

Suitable for 9th to 12th Grade

4	3	2	1
Demonstrates a thorough understanding when interpreting graphs of quadratic functions.	Demonstrates an understanding interpreting graphs of quadratic functions.	Demonstrates a partial understanding interpreting graphs of quadratic functions.	Demonstrates little understanding interpreting graphs of quadratic functions.
Very capably and independently manipulates algebraic expressions as they relate to quadratic functions.	Independently manipulates algebraic expressions as they relate to quadratic functions.	With some assistance, manipulates algebraic expressions as they relate to quadratic functions.	With limited accuracy manipulates algebraic expressions as they relate to quadratic functions.
Independently determines the relationships between the graphs and the equations of quadratic functions.	Determines the relationships between the graphs and the equations of quadratic functions.	Some effectiveness evident when determining the relationships between the graphs and the equations of quadratic functions.	Requires assistance to determine the relationships between the graphs and the equations of quadratic functions.
With complete accuracy, factors polynomials using the common factors, factors the difference of squares and factors trinomials of the form $x^2 + bx + c$.	With considerable accuracy, factors polynomials using the common factors, factors the difference of squares and factors trinomials of the form $x^2 + bx + c$.	With some accuracy factors polynomials using the common factors, factors the difference of squares and factors trinomials of the form $x^2 + bx + c$.	With minimal accuracy, yet some understanding, factors polynomials using the common factors, factors the difference of squares and factors trinomials of the form $x^2 + bx + c$.

PRACTICE #1: Make sense of problems and persevere in solving them.

- What constitutes a cognitively demanding task?

Lower-level demands (memorization)

- Involve either reproducing previously learned facts, rules, formulas, or definitions or committing facts, rules, formulas or definitions to memory.
- Cannot be solved using procedures because a procedure does not exist or because the time frame in which the task is being completed is too short to use a procedure.
- Are not ambiguous. Such tasks involve the exact reproduction of previously seen material, and what is to be reproduced is clearly and directly stated.
- Have no connection to the concepts or meaning that underlie the facts, rules, formulas, or definitions being learned or reproduced.

Lower-level demands (procedures without connections to meaning)

- Are algorithmic. Use of the procedure either is specifically called for or is evident from prior instruction, experience, or placement of the task.
- Require limited cognitive demand for successful completion. Little ambiguity exists about what needs to be done and how to do it.
- Have no connection to the concepts or meaning that underlie the procedure being used.
- Are focused on producing correct answers instead of on developing mathematical understanding.
- Require no explanations or explanations that focus solely on describing the procedure that was used.

Higher-level demands (procedures with connections to meaning)

- Focus students' attention on the use of procedures for the purpose of developing deeper levels of understanding of mathematical concepts and ideas.
- Suggest explicitly or implicitly pathways to follow that are broad general procedures that have close connections to underlying conceptual ideas as opposed to narrow algorithms that are opaque with respect to underlying concepts.
- Usually are represented in multiple ways, such as visual diagrams, manipulatives, symbols, and problem situations. Making connections among multiple representations helps develop meaning.
- Require some degree of cognitive effort. Although general procedures may be followed, they cannot be followed mindlessly. Students need to engage with conceptual ideas that underlie the procedures to complete the task successfully and that develop understanding.

Higher-level demands (doing mathematics)

- Require complex and non-algorithmic thinking – a predictable, well-rehearsed approach or pathway is not explicitly suggested by the task, task instructions, or a worked-out example.
- Require students to explore and understand the nature of mathematical concepts, processes, or relationships.
- Demand self-monitoring or self-regulation of one's own cognitive processes.
- Require students to access relevant knowledge and experiences and make appropriate use of them in working through the task.
- Require considerable cognitive effort and may involve some level of anxiety for the student because of the unpredictable nature of the solution process required.

Arbaugh, F., & Brown, C. A. (2005). Analyzing mathematical tasks: a catalyst for change? *Journal of Mathematics Teacher Education , 8*, p. 530.

- What can I do in my classroom to attend to multiple entry points, different solutions paths, and appropriate time? How do these features support student learning?
  - If a task meets the features of high cognitive demand (doing mathematics or procedures with connection to meaning), it should inherently have multiple entry points and solution paths.
  - The following journal article summarizes a study where tasks were well developed but problems with implementation reduced the cognitive demand of the task (one of the biggest factors was poor time allotment):
    - Stein, M. K., & Henningsen, M. (1997). Mathematical tasks and student cognition: classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education , 28* (5), 524-549.
  - Stein also has a text for teachers and teacher educators interested in synthesizing their current practice with new mathematics standards. Presented are cases of mathematics instruction drawn from research of nearly 500 classroom lessons. Readers will gain insight about how to foster a challenging, cognitively rich, and exciting classroom climate that propels students toward a richer understanding of mathematics.
    - Stein, M. K. (2000). *Implementing Standards-Based Instruction: A Casebook for Professional Development*. New York: Teachers College Press.

- Plan task-specific questions to explicitly integrate meta-cognition. The following list is adapted from
  http://www.scribd.com/doc/23035034/Metacognitive-questions
  - Reflective and Reasoning Questions:
    - How did you decide what to include?
    - Why did you write that/put that there?
    - How did you start off?
    - What did you find the most difficult? How did you tackle it?
    - Did you use any images in your head to help you?
    - How did you work together? Did it help?
    - How did you decide to leave information out?
    - What assumptions have you made?
    - What connections have you made? What makes a good connection?
    - Did you have a plan and did you have to change it?
    - Has anyone got an answer you like? Why?
  - Extension Questions:
    - Are features of this problem more important than others?
    - Where could you use what you have learned today with previous problems we have looked at?
    - What would be a different situation where your solution path would also work?

- An additional list of effective questions for mathematical thinking, developed by PBS TeacherLine, can be found by following the link below: http://mason.gmu.edu/~jsuh4/teaching/resources/questionsheet_color.pdf

- The chart below provides different types of meta-cognitive questions; some of these could be appropriate for students to ask one another as well (copied from http://mason.gmu.edu/~jsuh4/teaching/resources/mathjournals.pdf):

Reflecting on Problem Solving Clear Communication	**Respectful Communication**	**Flexible Thinking**	**Persistence**
What math words could help us share our thinking about this problem? Choose 2 and explain what they mean in your own words.	Did someone else solve the problem in a way you had not thought of? Explain what you learned by listening to a classmate.	What other problems or math topics does this remind you of? Explain your connection.	What did you do if you got "stuck" or felt frustrated?
What could you use *besides words* to show how to solve the problem? Explain how this representation would help someone understand.	Did you ask for help or offer to help a classmate? Explain how working together helped solve the problem.	Briefly describe at least 2 ways to solve the problem. Which is easier for you?	What helped you try your best? *or* What do you need to change so that you can try your best next time?
If you needed to make your work easier for someone else to understand, what would you change?	What helped you share and listen respectfully when we discussed the problem? *or* What do you need to change so that you can share and listen respectfully next time?	What strategies did you use that you think will be helpful again for future problems?	Do you feel more or less confident about math after trying this problem? Explain why.

PRACTICE #2: Reason abstractly and quantitatively.

- What is a *realistic* context?
  - We purposefully do not use the term *real-world* here because it is difficult to have a truly real-world context that can be reduced to something appropriate for the mathematics; thus, in saying *realistic* we mean that the situation can be imagined – a student could place themselves in the context

- NCTM describes key elements of reasoning and sense making specific to functions and representations (NCTM. (2009). Reasoning with Functions. In *Focus in High School Mathematics-Reasoning and Sense Making* (pp. 41-53). Reston: National Council of Teachers of Mathematics).
  - Representing functions in various ways – including tabular, graphic, symbolic (explicit and recursive), visual, and verbal
  - Making decisions about which representations are most helpful in problem-solving circumstances
  - Moving flexibly among those representations

- NCTM breaks down the knowledge of functions into five essential understandings. The fifth essential understanding pertains to multiple representations. The descriptors given below are major areas of focus under Big Idea 5. These essential understandings further describe what it means for a student to have flexibility with representations when studying functions (found in NCTM. (2010). *Developing Essential Understanding of Functions Grades 9-12.* Reston: National Council of Teachers of Mathematics).
  - <u>Essential Understanding 5a</u>. Functions can be represented in various ways, including through algebraic means (e.g., equations), graphs, word descriptions and tables.
  - <u>Essential Understanding 5b</u>. Changing the way that a function is represented (e.g., algebraically, with a graph, in words, or with a table) does not change the function, although different representations highlight different characteristics, and some may show only part of the function.
  - <u>Essential Understanding 5c</u>. Some representations of a function may be more useful than others, depending on the context.
  - <u>Essential Understanding 5d</u>. Links between algebraic and graphical representations of functions are especially important in studying relationships and change.

- Robin Rider gave instructional and assessment recommendations for teachers to improve student fluency with representations (Rider, R. (2007). Shifting from Traditional to Nontraditional Teaching Practices using multiple representations. *Mathematics Teacher , 100* (7), 494-500).
  - Vary the representation with introduction of new concepts – don't always present using the symbolic form… change between tabular, graphical, or symbolic
  - Create class discussion around strengths and weaknesses of different representations
  - Integrate different representations on assessments
  - Discussion around representation should include exploration of invariance
  - Use technology for further exploration of representations in a less tedious way

- Knuth looked at the specific struggles of students to connect and use different representations. Knuth's observations, given below, could help a teacher look for student misconceptions (Knuth, E. J. (2000). Understanding Connections between Equations and Graphs. *The Mathematics Teacher , 93* (1), 48-53).
  - Even when a graphical approach was significantly more efficient, the majority of students chose algebraic methods
  - When prompted to describe an additional solution process, many students were unable to recognize graphs as a viable path to a solution (17% of students gave an alternative solution method)
  - Students were unable to verify solutions using graphs; in fact, most students did not see the graphs as relevant at all in answering the questions
  - Students seem to have developed a ritualistic approach to finding solutions algebraically
  - Students can really only move in one direction with representations: from equation to graph

- The chart shown below could help in determining the appropriateness of different representations, as well as what features to attend to when posing questions to students (Friedlander, A., & Tabach, M. (2001). Promoting multiple representations in algebra. In NCTM, *The Roles of Representation in School Mathematics* (pp. 173-185). Reston: The National Council of Teachers of Mathematics, Inc).

Representation	Uses	Advantages	Disadvantages
Verbal	▪ problem posing ▪ interpretation of solution	▪ tool for solving problems ▪ connects mathematics and other domains	▪ language use can be ambiguous or misleading ▪ less universal
Numerical	▪ determine numbers to understand problem	▪ gives entry to a problem ▪ helps to investigate cases	▪ no generality ▪ some important aspects of problem or solution may not be visible
Graphical	▪ provide picture for a function of a variable	▪ visual ▪ intuitive for students	▪ lacks accuracy ▪ does not include all parts of domain and range
Algebraic	▪ general representation of a pattern	▪ able to manipulate	▪ can obstruct meaning or nature of the problem ▪ does not lend to interpretation of results

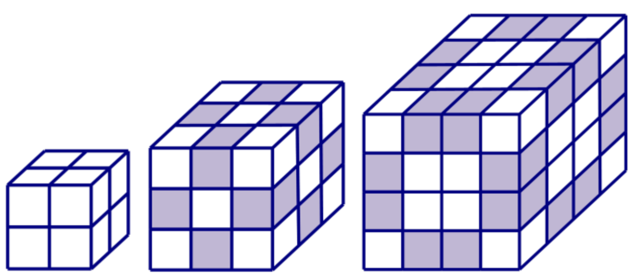
- The following three pages give examples of tasks involving developing and linking representations, as well as articulating connections.

PAINTED CUBES

(Adapted from Driscoll, M. (1999). *Fostering Algebraic Thinking.* Portsmouth: Heinemann.)

Each larger cube below is made up of smaller cubes. Someone decided to create a pattern by painting the smaller cubes that have ONLY two exposed faces in a different color. Examine the pattern and work together to complete each of the following tasks. You should use this paper to record notes and ideas; however, you will work as a group to create a poster with all of the information described below.

1. Discuss how many smaller cubes will be painted in the different color for the next larger cube.
2. Create a table relating the cube # to the total # of cubes with two faces painted in a different color.
3. Determine an equation relating the cube # to the total # of cubes with two faces painted in a different color; write your equation using function notation and explain how you arrived at your equation.
4. Create a graph showing the relationship between the cube # to the total # of cubes with two faces painted in a different color.
5. Select one coordinate pair from your table. Identify the same coordinate pair in the figures, the equation, and the graph.
6. Explain the relationship between the table, equation, and graph.



COMPARING SAVINGS PLANS

(Adapted from Coulombe, W. N., & Berenson, S. B. (2001). Representations of patterns and functions - tools for learning. In NCTM, *The Roles of Representation in School Mathematics* pp. 166-172). Reston: The National Council of Teachers of Mathematics, Inc)

Many high school students work during the summer and put some of their money into savings. This is very helpful when you go away to college and need some spending money. The easiest way to save money is to put a little bit away each week. Four different students - Brittni, Steven, Kyler, and Erik - have different savings plans. Review each plan below and then answer the questions that follow.

BRITTNI

Brittni had some money in savings from last summer. This past summer she put away a little bit of money each week. The table below shows how much Brittni had in her savings account after a given number of weeks.

week	1	2	3	4	5	6
Deposit Amount	250	290	330	370	410	450

ERIK

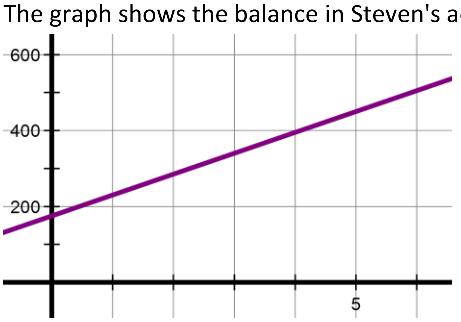
Erik's job automatically deposits a designated amount into his account every week but he can't remember how much. If the balance on his account is $B(x)$ where x is in weeks, Erik does know on two different occasions what the balance was: $B(4) = 550$ and $B(7) = 730$.

KYLER

Kyler's balance, $B(x)$, in his savings account can be represented $B(x) = 625 - 18x$ where x stands for the number of weeks.

STEVEN

The graph shows the balance in Steven's account based on the number of weeks.



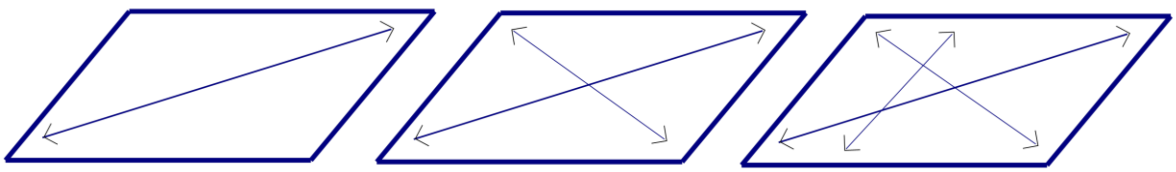
GROUP QUESTIONS

1. Which person do you think has the best savings plan? Justify your answer using an algebraic representation and a second representation of your choice.
2. How does each savings plan compare?
3. Who will have the most money in their savings account at the end of an 8 week summer? Explain how you know.

PATTERNS, PLANE AND SYMBOL

(NCTM. (2009). Reasoning with Functions. In *Focus in High School Mathematics-Reasoning and Sense Making* (pp. 41-53). Reston: National Council of Teachers of Mathematics)

Task: Develop a symbolic representation for a function that produces the number of regions in a plane formed by intersection lines such that no two lines are parallel and no more than two lines intersect in the same point, as shown in the figure.



1 line; 2 regions 2 lines; 4 regions 3 lines; 7 regions

Method 1: After exploring a number of cases students might produce a table of values for the number of lines and the number of regions. They can then use the table to develop a recursive definition of a function.

Method 2: Students could use a geometric approach with coins or tiles to create a pattern. Use the configuration of the pattern students might be able to determine the explicit form of the function.

Method 3: By applying technology to numeric and graphical reasoning, students may enter a number of ordered pairs from the table into a graphing calculator and examine a scatterplot of the pairs to the conjecture that the relationship is quadratic. Students could determine a regression equation and then test ordered pairs from the table.

Method 4: The teacher could ask students to focus on the differences between consecutive terms of the sequence of total regions. By applying algebraic reasoning, students may examine the data and observe that the function is quadratic because the first differences are linear so the second differences are constant. Then students could write a system of equations using the quadratic form and three ordered pairs.

PRACTICE #3: Construct viable arguments and critique the reasoning of others.

- The following links give information on constructing arguments
    - www.learner.org: This website walks through some concrete examples of how to introduce conjectures and simple proofs in class. The material is organized by grade levels and includes general numerical conjectures as well as geometric conjectures. The site also provides reflection questions for the teacher in thinking about the strategies of instruction and evaluation.
        - Grades 3-5: http://www.learner.org/courses/teachingmath/grades3_5/session_04/index.html
        - Grades 6-8: http://www.learner.org/courses/teachingmath/grades6_8/session_04/index.html
        - Grades 9-12: http://www.learner.org/courses/teachingmath/grades9_12/session_04/index.html

    - Henri Picciotto has generously put up a lot of resources on the web regarding his school's Geometry course, specifically on the development of conjectures and proof reasoning. The site links to the three complete Geometry units that appear to be well-written and very thoughtful.
        - Geometry proofs classroom material: http://www.mathedpage.org/proof/math-2.html

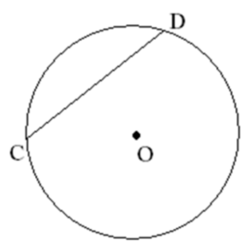
    - Von Hiele levels are a series of stages in the development of formal geometric logical arguments. This group of links collectively how teachers can implement geometry lessons while keeping in mind students' cognitive learning needs.
        - How to teach geometry proofs: http://www.homeschoolmath.net/teaching/geometry.php
        - Illustration of Von Hiele levels: http://math.youngzones.org/van_hiele.html
        - Specific examples: http://investigations.terc.edu/library/bookpapers/geometryand_proof.cfm

- Here is an interesting strategy for increasing student-to-student dialogue within the classroom, via the implementation of "dialogue journals" that get exchanged between students for written feedback.
  - Implementation of dialogue journals: http://www.teachervision.fen.com/writing/skill-builder/48885.html
  - Potential benefits of dialogue journals: http://www.ucalgary.ca/iejll/Joyce+Bainbridge

- Teaching students to recognize solid vs. weak arguments is a challenging task. The resources below address how to evaluate student arguments.
  - Habits of Mind: http://www2.edc.org/cme/showcase/HabitsOfMind.pdf (Pg. 11 - 12)
  - *CME Project* textbooks from Educational Development Center (EDC) model a series of "episodes" of conversations between students throughout the books that expose readers to new concepts through proof-like reasoning constructed in student-friendly language. An example, which could be used for classroom discussion, is given below:

Minds in Action – Episode 17

Looking at the definition of diameter, Tony wonders about chords.



Tony I once heard that the diameter is the longest chord you can draw for a given circle. Did you know that, Sasha?

Sasha As a matter of fact, I did! I think I can prove it. Let's see.

Tony Well, make life simple and start with the first circle at the bottom of the last page. You already have a chord, *CD*.

Sasha Right! All we have to do is look for triangles, and I love triangles! Connect *C, D,* and *O.* Now I remember that in a triangle the sum of two sides is always greater than the third one. So $CD < CO + OD$.

Tony You're brilliant! I know what to do now. I just noticed that *CO* and *OD* are two radii, so their sum is equal to the diameter. So we've proven that any chord is shorter than a diameter.

Benson, Jean, et. al. (2009). Geometry. *CME Project,* 392-393.

- Here is an example of a challenging problem that pushes student thinking and argument-formation. In the process of arriving at the solution, students should be using a mixture of deduction and experimentation. In the end, the proof is based largely on students' construction, and students should be confident of their results (taken from http://jwilson.coe.uga.edu/Situations/Framework.Jan08/articles/Cuoco1996HabitsofMind.pdf).
    - *A square birthday cake is frosted on top and on the four sides. How should it be cut for 7 people if everyone is to get the same amount of cake and the same amount of frosting?*

PRACTICE #4: Model with mathematics

- What does it mean to model with mathematics?
    - In the document "Habits of Mind" by Al Cuoco, et. al. (http://www2.edc.org/cme/showcase/HabitsOfMind.pdf), mathematical modeling is defined as the attempt to abstract specific mathematical cases into a more general understanding of the concept or behavior.
        - "Getting good at building and applying abstract theories and models comes from immersion in a motley of experiences; noticing that the sum of two squares problem connects to the Gaussian integers comes from playing with arithmetic in both the ordinary integers and in the complex numbers *and* from the habit of looking for similarities in seemingly different situations. But, experience, all by itself, doesn't do it for most students. They need explicit help in what connections to look for, in how to get started."

        - This same document outlines techniques for building abstractions and models, which are techniques that should be taught explicitly as mathematical habits to students who are learning to model.
            - ◊ Model objects and changes with functions
            - ◊ Look for multiple perspectives (graphical, algebraic, arithmetic) and to find ways to combine those perspectives to reach deeper conclusions and connections.
            - ◊ Mix deduction and experimentation.

- Within the Common Core standards, modeling is described as follows (see http://www.corestandards.org/the-standards/mathematics/high-school-modeling/introduction/ for more information):
    - "Modeling links classroom mathematics and statistics to everyday life, work, and decision-making. Modeling is the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions. Quantities and their relationships in physical, economic, public policy, social, and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and comparing predictions with data."

- The Common Core Standards give 6 stages within a complete cycle of modeling:
    1) Identifying variables in the situation and selecting those that represent essential features,
    2) Formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between the variables,
    3) Analyzing and performing operations on these relationships to draw conclusions,
    4) Interpreting the results of the mathematics in terms of the original situation,
    5) Validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable,
    6) Reporting on the conclusions and the reasoning behind them.

- These tips pulled from http://serc.carleton.edu/introgeo/models/HowToUse.html are for introducing mathematical modeling in a science class, but they are useful tips for general mathematical modeling of physical situations.
    - Keep the activity as interactive as possible. When you find that you're spending a majority of your time lecturing to the students about what to do or how things work, try to think of ways you can get them working through ideas in groups, lab, interactive lectures, etc.

    - Including students in the development process and/or providing opportunities for them to experiment with the model or modify it can increase students' understanding of the model and its relationship to the physical world.

    - Creating opportunities for students to analyze and comment on the models behavior increases their understanding of the relationships between different inputs and rates.

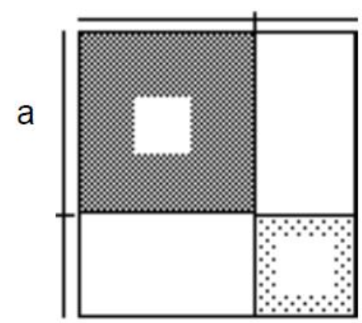
    - Creating opportunities for students to validate the model, i.e. compare model predictions to observations, increases their understanding of its limits.

    - Stress that models are not reality and that a model's purpose is to help bridge the gap between observations and the real world. An important reason to use a model is that you can perform experiments with models without harming the system of interest.

- Make sure that students think about the underlying assumptions of a model and the domain of applicability. Try to ask questions that can help check their understanding. For example, simple exponential growth assumes that the percent growth rate remains fixed and in real world systems it only applies for so long before the system becomes overstressed. Having students identify underlying assumptions of a model and their domain of applicability can help them gain an appreciation of what a model can and cannot do.

- Models can be used to explore "What-if" scenarios. "What if Atmospheric CO2 doubles?" is a common example for a climate model.

- A key part of modeling is visualization. Here is an example for geometric modeling of an algebraic concept, as taken from Pg. 7 of http://www2.edc.org/cme/showcase/HabitsOfMind.pdf.



$(a+b)^2 = a^2 + 2ab + b^2$

- Other examples of modeling, as provided by the Common Core outline, include:
  - Estimating how much water and food is needed for emergency relief in a devastated city of 3 million people, and how it might be distributed.
  - Planning a table tennis tournament for 7 players at a club with 4 tables, where each player plays against each other player.
  - Designing the layout of the stalls in a school fair so as to raise as much money as possible.
  - Analyzing stopping distance for a car.
  - Modeling savings account balance, bacterial colony growth, or investment growth.
  - Engaging in critical path analysis, e.g., applied to turnaround of an aircraft at an airport.
  - Analyzing risk in situations such as extreme sports, pandemics, and terrorism.
  - Relating population statistics to individual predictions.

- EDC has provided some additional information about modeling here: http://thinkmath.edc.org/index.php/Model_with_mathematics

- For ideas of problems or real-world situations that involve mathematical modeling, see http://www.math.montana.edu/frankw/ccp/modeling/topic.htm and http://www.indiana.edu/~iucme/mathmodeling/lessons.htm. The former connects math to various physical phenomena, and the latter is a collection of modeling math lessons that teachers have used inside their classrooms.

PRACTICE #5: Use appropriate tools strategically

- There are many online manipulatives and games that can be used to support a task or to gain a deeper understanding of functions.
  - NCTM Illuminations: This website has many resources and lesson ideas that allow for the integration of various teaching tools into functions lessons. http://illuminations.nctm.org/
  - National Library of Virtual Manipulatives: Features online manipulatives that can be used as learning tools. http://nlvm.usu.edu/en/nav/vlibrary.html
  - Smart Skies: This game was developed by NASA to help students with their understanding of linear functions. http://www.smartskies.nasa.gov/
  - Equations of Attack: In this game, students try to sink their opponent's ship using their knowledge of linear functions. http://illuminations.nctm.org/LessonDetail.aspx?id=L782
  - Exploring Linear Functions: This lesson includes an online manipulative that allows students to systematically change the slope and y-intercept of a linear function and to observe patterns that follow from their changes. http://www.nctm.org/standards/content.aspx?id=26790
  - Function Matching: Students can use this manipulative to see if they can produce equations that match up with a graph of a given function. http://illuminations.nctm.org/ActivityDetail.aspx?ID=215

- The information below about estimation is from http://arb.nzcer.org.nz/supportmaterials/maths/concept_map_estimation.php.
  - When do we estimate? Students need to be taught how and when to estimate effectively. There are three cases in which estimation is useful:
    - There is no need to have an exact answer. An estimate is good enough: for example *"Do I have enough money?"*
    - There is not enough information to get an exact answer: for example, *"About how many times will my heart beat in an hour?"*
    - To check if the answer from a calculation is sensible.

  - What are some strategies of estimation?
    - **Reformulation,** which changes the numbers that are used to ones that are easy and quick to work with.
    - **Compensation,** which makes adjustments that lead to closer estimates. These may be done during or after the initial estimation.

- **Translation,** which changes the mathematical structure of the problem (e.g. from addition to multiplication). Changing the form of numbers so that it alters the mathematical structure of the problem is also translation (Example: 26.7% of $60 requires multiplication, but this is about 1/4 of $60, which uses division).

- This site provides examples on how to teach estimation in early grades: http://www.nsa.gov/academia/_files/collected_learning/elementary/arithmetic/reasonable_estimates.pdf. At the secondary level, besides using estimation to check answers, students regularly have trouble graphing an estimated location of a "nasty" point in the coordinate plane. Teachers at earlier grades can address this issue by asking students to plot points along estimated locations on a number line.

- This outline from a talk provides a nice visual for why estimation is useful within the reasonableness framework. It also provides some concise ideas on how to structure activities in a way that encourages students to estimate. http://www.nesacenter.org/uploaded/conferences/SEC/2011/handouts_teachers/Hanania_handout.pdf

PRACTICE #6: Attend to precision

- Resources that focus on precision within functions:
    - Green Globs: This game requires students to destroy globs by creating functions that pass through them. http://www.greenglobs.net/
    - Exploring Linear Functions Exploring Linear Functions: This lesson includes an online manipulative that allows students to systematically change the slope and y-intercept of a linear function and to observe patterns that follow from their changes. http://www.nctm.org/standards/content.aspx?id=26790
    - Precise Mathematical Language: This article addresses precision as students communicate with one another. http://scimath.unl.edu/MIM/files/research/KrandaJ.pdf
    - Pi Filling, Archimedes Style: Students can explore how precision affects their results as they use this method of finding the digits of pi. http://illuminations.nctm.org/LessonDetail.aspx?id=L714

- Some cautions for attending to precision follow. These could be helpful guidelines for students (found at http://www.college-algebra.com/essays/writing_mathematics_correctly.htm):
    - Use complete sentences, correct grammar, and correct spelling.
    - Symbols (like the + symbol) that have a specific mathematical meaning are reserved for mathematical use.
    - Many mathematical adjectives and nouns have precise mathematical meanings, and an English synonym will not serve as a replacement. For example, "element" and "part" are not interchangeable when referring to an element of a set.
    - Look at examples of writing in the textbook, and try to emulate the style.
    - Two mathematical expressions or formulas in a sentence should be separated by more than just a space or by punctuation; use at least one word.
    - Words have meanings: be aware of them. For example, an ***equation*** has an equal sign in it. An ***expression*** is an algebraic combination of terms with no equal sign.
    - Don't use abbreviations.
    - Don't end a line with an equal sign or an inequality sign.
    - Honor the equal sign.
    - Use different letters for different things.
    - Define all terms or variables.

  - Once a variable has been assigned a meaning, do not re-use it with a different meaning in the same context.
  - Avoid the use of imprecise terms.
  - Be sure that the use of a term agrees with the definition of that term.
  - Conclude the solution of a problem with a clear and complete statement of the conclusion.

- Professor Hung-Hsi Wu from Berkeley's mathematics department has written copiously on the issue of teacher precision in the classroom. He considers Precision one of the basic characteristics of the essence of mathematics that is important for K-12 teaching, along with Definitions, Reasoning, Coherence, and Purposefulness. He defines precision as "Mathematical statements are clear and unambiguous. At any moment, it is clear what is known and what is not known," and provides specific examples of opportunities where teachers can ask probing questions to attend to precision. Wu asserts that "Precision in the vocabulary is necessary because it is only through this vocabulary that we can transcribe intuitive spatial information into precise mathematics, and it is entirely on this vocabulary that we base our reasoning."

  Some issues Wu brings up include: definition of similarity/congruence; meaning of the equal sign; and recognizing when estimation or simplifying assumptions are necessary. See http://math.berkeley.edu/~wu/schoolmathematics1.pdf for more info.

- Precision in communication can start with pictoral representations such as this one: http://www.learner.org/courses/teachingmath/grades6_8/session_02/section_02_b.html . In order for students to respond effectively to the pictured prompt, they have to employ precise language and to explain their thoughts either orally or in written language, possibly with the aid of symbols. Clearly, students cannot practice precision if they lack opportunities to communicate inside the classroom. Some practical strategies for creating opportunities for communication within the math classroom are outlined at http://www.learner.org/courses/teachingmath/grades6_8/session_02/section_03_d.html,
  http://www.ascd.org/ASCD/pdf/journals/ed_lead/el_198509_eaton.pdf, and
  http://www.edu.gov.on.ca/eng/literacynumeracy/inspire/research/CBS_Communication_Mathematics.pdf.

- Here are some suggested criteria for characteristics of "good" writing in mathematics. This list is intended for teachers to use in grading student assignments (See http://www3.saintmarys.edu/departments/mathematics-computer-science/department-policies/advanced-writing-requirement-mathematics/criteria-good-writing):

    1. **Accuracy**: The paper is free of mathematical errors, and the writing conforms to good practice in the use of language, notation, and symbols.
    2. **Organization**: The paper is organized around a central idea. There is a logical and smooth progression of the content and a cohesive paragraph structure.
    3. **Clarity**: Explanations of mathematical concepts and examples are easily understood by the intended audience. The reader can readily follow the paper's development.
    4. **Insight**: The paper demonstrates originality, depth, and independent thought.
    5. **Mechanics**: The paper is free of grammatical, typographical, and spelling errors. The mathematical content is formatted and referenced appropriately.

PRACTICE #7: Look for and make use of structure

- What does it mean to look for and make use of structure?
  - Students can look at problems and think about them in an unconventional way that demonstrates a deeper understanding of the mathematical structure – leading to a more efficient means to solving the problem.
  - Example problems from Gail Burrill:
    - Solve for x: $3(x-2)=9$
      - ◊ Rather than approach the problem above by distributing or dividing, a student who uses structure would identify that the equation is saying 3 times something is 9 and thus the quantity in parenthesis must be 3.
    - Solve for x: $\frac{3}{x-1}=\frac{6}{x+3}$
      - ◊ The "typical" approach to the above problem would be to cross multiply and solve; a student who identifies and makes use of structure sees that the left side can be multiplied by 2 to create equivalent numerators… then simply set the denominators equal and solve.

  - Many examples of a students' ability to identify and make use of structure show up in the SAT – the example problems that follow are best approached from a structural point of view rather than a strictly procedural approach
    - Given $a-b=3$ and $a^2-b^2=12$, what does $a+b=$?
      - ◊ In this example students have to identify the structure of a difference of squares and see the connection between the givens and the question.
    - If $3x-2y=10$ and $x+3y=6$, what is $4x+y$?
      - ◊ In this example students have to identify the structure of a system of equations… rather than solve for each variable, the most efficient approach is to look at the connection between the givens and the question and identify that the two equations can be added to produce $4x+y$.

- A sequence begins with the number 5. The next term is found by adding 4, and the next term is found by multiplying by -1. If this pattern of adding 4 and multiplying by -1 continues, what is the 26th term in the sequence?
    - ◊ In this example students have to identify a pattern by grouping… the sequence is 5, 9, -9, -5, 5, 9, -9, … they can then identify the pattern repeats after groups of 4… students can then use this to see how many groups of 4 are in 26 and use the remainder to identify the correct term.

PRACTICE #8: Look for and express regularity in repeated reasoning

- http://mason.gmu.edu/~jsuh4/impact/PFAEntire.pdf Patterns, Functions, and Algebra for Elementary School Teachers: This is a professional development training that focuses on getting students to generalize processes and develop algebraic thinking at the elementary school level.

- http://www.nctm.org/eresources/article_summary.asp?URI=JRME2011-07-308a&from=B Generalizing-Promoting Actions: How Classroom Collaborations Can Support Students' Mathematical Generalizations. This article focuses on how teachers can help students to develop mathematical generalizations through collaboration.

- http://ed-osprey.gsu.edu/ojs/index.php/JUME/article/view/100/81 Forging Mathematical Relationships in Inquiry-Based Classrooms with Pasifika Students: This article presents research on how students develop mathematical practices and competency when placed in classrooms that focus on inquiry and finding patterns.

- http://illuminations.nctm.org/LessonDetail.aspx?id=L646 Barbie Bungee: In this activity students record data as they simulate a bungee jump with a Barbie doll. They are prompted to make generalizations about their data they find to report a general pattern.

- Example problem involving generalization and expressing regularity from Cuoco, Al, et. al. (2009). Algebra 2. *CME Project,* p. 148:
  a) If a is some fixed number, find the general form of $(x – a)(x^2 + ax + a^2)$.
  b) If a is some fixed number, find the general form of $(x – a)(x^3 + ax^2 + a^2x + a^3)$.
  c) What is a general result suggested by parts (a) and (b)?

- http://www2.edc.org/mathpartners/pdfs/6-8%20Patterns%20and%20Functions.pdf Patterns and Functions: This resource from the EDC outlines many different activities and classroom resources that are centered around pattern recognition within functions.

- http://www.learner.org/courses/learningmath/algebra/ Patterns, Functions, and Algebra: This is a series of Professional Development videos that focus on helping students to find patterns and develop structures within major algebra topics.

- http://ncsmonline.org/docs/resources/journals/NCSMJournalVol12Num1.pdf Prediction as an Instructional Strategy: This article from NCSM discusses how to use prediction as an instructional strategy to support students as they look for structures and connections in mathematics.

- http://illuminations.nctm.org/LessonDetail.aspx?id=L658 Golden Ratio: This lesson from NCTM is structured so that students make connections between the golden ratio and Fibonacci numbers. It requires them to record information and then look for patterns and structures within their recordings.

Problem-Solving Rubric

(Adapted from Dept. of Chemical Engineering CRCD Project, August 2002)

Criteria	Exemplary	Good	Needs Improvement
Identifying problem and main objective	**(4 – 5)**	**(2 – 3)**	**(0 – 1)**
Initial questions	Questions are probing and help clarify facts, concepts, and relationships in regard to problem. Follow-up questions are gleaned from appropriate sources.	All questions may not be relevant. May have some difficulty formulating questions to move toward better understanding of the problem.	Few or not questions formulated. Expects others to define the questions. Does not seem to understand the central problem.
Understanding the problem	Clearly defines the problem and outlines necessary objectives in an efficient manner.	Problem statement has some ambiguity or misses some important issues.	Problem is defined incorrectly or too narrowly. Key information is missing or incorrect.
Seeking information	Identifies several sources of information and individuals for support.	Relies on a few sources only. Does not gather extensive information.	Not clear as to what is needed. Waits to be told. Does not seek information sources.
Applying previous knowledge			
Integration of knowledge	Effectively applies previous knowledge to current problem. Integrates with new information to assist problem solving process.	Applies limited amount of prior knowledge to current problem. Does not consistently use information effectively.	Unable to make connection to previous knowledge. Unwilling to review summaries of prior knowledge for useful information.
Sharing previous knowledge	Team members all work together to gain knowledge and apply and synthesize information. All listen respectfully to the opinions of others.	Some exchange of information and discussion occurs, but team members do not work consistently to address each one's needs or understanding.	Each team member must teach him/her self. No sharing of knowledge among team.
Identifying information			
Use of information	Consistently gathers a broad spectrum of resources and information and integrates it with prior knowledge and problem-solving strategies.	Information gathered may not be extensive, or may have occasional difficulty using information effectively in problem solving.	Fails to gather information, or obtains it from limited or inappropriate sources. Can't make connection between information gathered and the problem.

Criteria	Exemplary	Good	Needs Improvement
Framework	Creates and applies a framework (e.g. diagram, written description) throughout the process. Revises it as necessary.	Can create a framework but may not use it consistently in an effective manner, or revise it as needed.	Creates a vague framework that doesn't move the problem-solving process along. Doesn't seek help from others.
Tasks	Team takes the initiative to define tasks, match assignments to expertise, rotate responsibilities, maintain open communication, and develop strategies to enhance group success.	All team members generally cooperate and prioritize tasks, but may not consistently rotate responsibilities or work out most effective strategies for success.	Team spends time on tasks that interfere with the problem-solving process. Team members don't know who is responsible for which task.
Designing and conducting experiments			
Design	Each team member can describe planned experiments and how they relate to the problem; relate hypotheses to previous knowledge; identify necessary steps and timeline for project.	Description of planned experiments, relation of hypotheses, identification of steps and timeline, can be accomplished by joint effort of the whole team but not by each team member.	Fails to formulate hypotheses to test. Does not express possible outcomes.
Use of evidence	Continuously uses results to refine plan. Draws correct conclusions from results. Generates appropriate visual aids that facilitate understanding of the problem. Explores new ways to approach problem.	Usually adjusts experimental plan on basis of new knowledge. Usually plots/tabulates results to aid in reaching conclusions.	Data obtained are inadequate or incorrectly calculated. Tables and graphs are not prepared or are difficult to read and interpret. Conclusions are incorrect or not based on evidence.
Documentation	Comprehensive collection of raw and summarized data. Includes detailed information to allow repetition of experiments based only on written notes.	Data are summarized and organized, but may lack some details or some explanation necessary for repetition of experiments.	Laboratory notes aren't organized. Experimental results cannot be easily found. Experiments cannot be repeated because of lack of information.
Analyzing and interpreting results			
Use of analytic tools	Consistently uses new procedures and tools	Uses new methods and tools, but may not	Errors made in analytical methods,

	successfully, and can describe rationale for them. Runs appropriate control and replicate experiments.	always be successful. May not accurately explain rationale. Control and replicate experiments run.	but sources of error aren't found. Appropriate control or replicate experiments not run.
Interpretation of data	Able to describe results and conclusions clearly and concisely. Relates results to hypothesis and to currently accepted theory.	Draws correct conclusions from results, but may not relate them well to original hypothesis or current theory.	States conclusions without justification. Does not consider internal consistency of results. Cannot compare control or replicate results.
Analyzing alternative interpretations and solutions	Can account for unexplained results. Recognizes limitations of current hypothesis and proposes alternative interpretations.	Recognizes results that don't fit hypothesis but may not readily come up with alternative interpretations.	Does not recognize that results do not conform to original hypothesis. Cannot suggest alternative interpretation.
Assessing self and others			
Problem solving process	Critically reflects on problem-solving techniques, strategies, and results. Identifies those most helpful to self. Offers clear insights regarding self-knowledge.	Can identify problem-solving techniques that are most helpful, but may not be able to clearly summarize self-knowledge.	Unable to reveal insights about own learning. Cannot discuss relevance of problem-solving techniques.
Collaborative learning	Group develops strategies for success, and demonstrates understanding of how problem solving process relates to other activities. Creates a positive environment for reflection on the learning process.	Group can assess the contributions of members' skills, knowledge, and attitudes to the success of the team, but may not develop an overall strategy for success or overview of problem solving.	Assessments of group performance are not insightful. No commitment to group skill development for the future is shown. Little or no attention paid to group morale.
Overall assessment	Clearly and concisely articulates the problem-solving process and describes how well it was applied to the current problem.	Can describe the problem-solving process, but may not critically assess how well it was applied to the current problem.	Shows little or no understanding of the problem solving process, and cannot assess how well it was applied to the current problem.

GENERIC MATH RUBRIC

4	3	2	1
Demonstrates a thorough understanding of the main concepts.	Demonstrates an understanding of the main concepts.	Demonstrates a partial understanding of the main concepts.	Demonstrates little understanding of the main concepts.
Well organized with correct answers.	Organized and most answers are correct.	Organization needs to improve, some correct answers.	Very weak evidence of organization, a few correct answers
Mathematical terms and symbols are used appropriately and are often elaborated upon.	Mathematical terms and symbols are used appropriately.	Some mathematical terms and symbols are used correctly.	Mathematical terms and symbol use are weak, not enough references to mathematical terms are used.
Thorough analysis of the problem with accurate solutions.	Analysis of the problem is evident, considerable accuracy.	Analyzes the problem with some success, accuracy needs to improve.	Very little evidence of analysis. Some educated guesses. Accuracy is weak.